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Research Article

Odd Prime Labelling of Mobius Ladder Graphs

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Abstract

Graph theory is a subfield of discrete mathematics that explores networks of points connected by lines. Graph theory is a significant area of mathematics with many applications in different domains. Graph labeling has been one of the most significant and intriguing areas of graph theory research. Depending on the needs, graph labeling involves giving integers to vertices, edges, or both. Giving the vertices of a corresponding graph distinct odd prime numbers is known as "odd prime labeling," and it is a more intricate and specialized branch of graph theory. This special labeling method can be used to describe the connections and characteristics of Mobius ladder graphs as well as to respond to concerns regarding isomorphism, network architecture, encryption, and number theory. Mobius ladder graphs with odd prime labeling present a fascinating method to study how vertices and prime numbers interact in this specific kind of graph structure. From the topological elegance of the Mobius ladder to the rhythmic harmony of prime differences, this idea unveils a wide range of mathematical relationships and insights. Graph theory and number theory combine to produce a masterwork on the Mobius ladder, an odd prime-labeled work of mathematical art. Here, we applied an odd prime labeling, a prime labeling version of Mobius ladder graphs. In this research, we also presented general proof that every Mobius ladder graph is an odd prime labeling graph, as well as a generalized method for using odd sequences to demonstrate that all Mobius ladder graphs are odd prime.

Keywords: *Greatest common divisor(gcd), Mobius Ladder graph, Odd sequences, Odd prime labeling, Relative prime*

1. Introduction

The Odd Prime Labeling of Mobius ladder graphs is a fascinating area of graph theory in which labels are assigned to the vertices according to odd numbers, and the adjacent vertices are odd prime numbers. S. Meena and P. Kavitha carried out further development of the concept of prime labeling, which was first put forth by Roger Etringer in 1980. K.P. Shah and U.M. Prajapati introduced the idea of odd prime labeling. S. Meena, P Kavitha, and G. Gajalakshmi are among the researchers who have since worked on the subject (Meena and Gajalakshmi, 2022). A particular class of graphs called Mobius ladder graphs is distinguished by its distinct structure, which is akin to a ladder with a twist and was inspired by the topological Mobius strip. To explore this subject further, it is essential to comprehend the basic components at play. Vertices and edges, which indicate relationships between entities and vertices, make up a graph. Mobius ladder graphs show the complex interaction between vertices and edges, adding an intriguing twist to conventional ladder structures (Redha *et al*., 2022). These graphs become even more complex when one considers the concept of "odd prime labeling". The process of giving labels to a graph's vertices in a way that ensures certain mathematical properties hold, usually integers, is known as odd prime labeling. Odd prime labeling, one of the most fundamental labeling methods, is labeling the vertices with a set of odd integers (1,3,5,...,2n-1), where n is any positive integer, such that any two adjacent vertices are relatively prime (Meena and Gajalashmi, 2022). Moreover, the symmetry of Mobius ladder graphs is one of its distinguishing features. They differ from conventional ladder graphs in that the ladder's twist introduces a special symmetry. When odd prime labeling is added to this symmetry, complex relationships between the labeled vertices result, making the combination especially fascinating. Examining cycles, paths, and connectivity within the structure of these graphs is common when studying them. The odd prime labeling of Mobius ladder graphs is an intriguing area of intersection between number theory and graph theory, exploring the complex interactions between prime numbers and vertices in this particular type of graph structure. In addition, with Mobius ladder graphs' unique labeling scheme, Mobius ladder graphs' structural properties can be better understood mathematically (Wannasiri Wannasit and Saad El-Zanati, 2017)

On the other hand, the study of the abstract relationships between objects in a mathematical structure called a graph is the focus of graph theory, considered a fundamental subject in discrete mathematics. Vertices, also referred to as nodes, and the edges that join them form a graph. This branch of mathematics has applications in biology, computer science, and problems with optimization problems, among other fields. In general terms, there are two different types of graphs: directed and undirected. While each edge in a directed graph has a specific direction from one vertex to another, edges in an undirected graph have no direction at all (Snehal *et al.,* 2024). When modeling relationships with a specific flow or order, this directional aspect is particularly beneficial as it adds a level of complexity. In a graph, edges can have different characteristics such as weights, which indicate how much it requires or how far it takes to go from one vertex to another. In numerous distinct optimization problems, where determining the minimum spanning tree or the shortest path becomes essential, weighted graphs are vital. A key component of Graph theory is the study of paths and cycles. A cycle is a closed path in which the beginning and ending vertices are the same. A path is a series of vertices where successive vertices are adjacent (Barbehenn, 1998). Special cases that visit every edge exactly once are Hamiltonian cycles and Eulerian paths, the latter of which necessitate distinct vertices and the former of which enables repeated vertices. In Graph theory, a crucial idea that characterizes a graph's cohesiveness is connectivity. Every pair of vertices in a connected graph has a path connecting them, whereas a disconnected graph consists of discrete elements without any path connecting them. When assessing the reliability of networks and communication effectiveness, this feature is essential. Graphs can be classified as either acyclic or cyclic based on the existence of cycles. Computer science makes substantial use of acyclic graphs, especially trees, for searching algorithms and hierarchical structures. Acyclic graphs, such as trees, are used in data structures, database indexing, and processes for making decisions (Arul *et al.,* 2023). Moreover, graphs can be bipartite, which means that two sets of vertices can be separated from one another by edges that only link vertices from the two sets. This feature can be used in modeling situations where there are relationships between different groups, like resource allocations or job assignments. Understanding graph embedding on surfaces necessitates an understanding of planar graphs. Any graph that can be drawn on a plane without any edges crossing is referred to as a planar graph. Understanding spatial relationships, circuit design, and geographical mapping all depend on this concept. Graph theory introduces various parameters to analyze graphs, such as degree, which represents the number of edges incident to a vertex. The degree sequence of a graph is a list of its vertices' degrees, providing insights into the graph's structure and connectivity (Mateusz *et al.*, 2019). Graph algorithms are essential for effectively resolving real-world issues. In a weighted graph, for instance, Dijkstra's algorithm determines the shortest path between two vertices, whereas Kruskal's algorithm effectively determines a minimum spanning tree. Allocating resources, designing networks, and optimizing routes are all based on these algorithms. A common tool used to model social networks and the World Wide Web is the graph. In the analysis of social networks, vertices stand in for individuals and edges for the connections between them. This abstraction makes it feasible for researchers to study community structures, determine important nodes, and analyze information flow. Graph theory is a crucial tool in discrete mathematics that helps to solve real-world problems. It is an important area of mathematics with applications spanning from computer science to biology and beyond since it offers a flexible framework for modeling relationships, structures, and processes. Graph theory will always be applicable for solving new problems and expanding our understanding of complex systems because of its continuing development (Rahman and Yen, 2023).

Furthermore, the next phase is looking at the nature of labeling and the different kinds of labeling in graph theory. Within the field of discrete mathematics called graph theory, there is a fascinating and complex area known as graph labeling. Fundamentally, graph labeling is the process of giving labels or numerals to the different components of a graph, for edges and vertices. A methodical investigation of the structural and combinatorial properties of graphs is made possible by the particular rules and properties that regulate this process. There are several approaches to the study of graph labeling, and each one provides a different perspective on the structure of graphs and how they are used in different fields (Dhokrat, 2024). Vertex labeling is one of the basic kinds of graph labeling. Every vertex in the graph is given a unique label or a label from a certain set when vertex labeling is applied. The main objective is to develop a systematic approach for labeling vertices that improves the graph's visualization and analysis. "Graceful labeling" is a significant subtype of vertex labeling, in which labels are assigned so that the resulting edge labels make up an arithmetic sequence. This enriches the graph's visual appeal and highlights interesting arithmetic characteristics hidden within its design. An additional subtype, "harmonic labeling", involves giving labels to the graph according to harmonic numbers, giving it a harmonic mathematical structure. Another important kind of graph labeling that focuses on labeling a graph's edges is called "edge labeling". In some cases, these labels represent weights, distances, or additional relevant quantities connected to the edges (Dushyant and Tanna, 2018). One popular type of edge labeling is “distance labeling”, which is labeling edges according to the distances between their endpoints. This is particularly helpful in applications that heavily rely on spatial relationships or geographic separations between connected vertices. Edge labeling becomes crucial in the context of weighted graphs to represent the weights attached to each edge. This aspect of graph labeling is essential for solving optimization problems, where the minimum spanning tree or shortest path is the main objective. By simultaneously assigning labels to vertices and edges, total graph labeling adopts a holistic approach. A well-known variation of total graph labeling is "total coloring". Vertices and edges in the total coloring are given colors so that no two adjacent elements have the same color. In doing so, the idea of graph coloring is expanded to encompass edges as well as vertices, offering a thorough comprehension of the chromatic properties of the graph (*Zeng et al., 2024*).1"Equitable total coloring" is another subtype that provides a balanced color distribution, assisting to develop a symmetrical and visually appealing representation of the graph. Graph labeling has multiple applications in many different fields. Graph labeling is a crucial concept in computer science as it assists in developing effective algorithms and data structures for graph processing. To comprehend information flow, connectivity, and centrality within networks, vertices, and edges need to be labeled systematically. Graph labeling is a useful tool in biology, where molecules are represented by vertices and bonds by edges in model molecular structures. These elements' labels shed light on the spatial organization and characteristics of molecules (Janusz and Dybizbański, 2023). Graph labeling is also important for optimization problems, particularly when routing, network design, or resource allocation must be done efficiently. The labels put on the edges of weighted graphs, which stand for costs or distances, help algorithms that aim to maximize these variables. Graph labeling continues to evolve as technology progresses, leading to new trends and uses. The need for sophisticated graph labeling strategies that can capture and analyze complex relationships is growing as real-world systems, like social networks and biological networks, become more complex. This area of study investigates new graph labeling strategies that address the difficulties presented by large-scale, dynamic networks (Dhokrat, 2024).

In addition to the labeling techniques already mentioned, odd prime labeling is an essential labeling method. Odd prime labeling is a unique form of graph labeling in graph theory where labels are assigned to vertices from the set of odd numbers, typically beginning from 1. When specific requirements or guidelines are met during the labeling process, the graph shows them in a unique and organized way. Odd prime labeling is the process of giving an odd number to every vertex in the graph with adjacent vertices that are relatively prime. Labeling usually begins with the smallest odd number, 1, and moves on to larger odd numbers for the remaining vertices. When two numbers have a greatest common divisor (gcd) of 1, they are said to be relatively prime. This condition adds a special constraint to the labeling process by providing that adjacent vertices have labels that do not share common factors other than 1. Odd prime labeling is a notion that is frequently used with particular graph classes, like trees or particular graph families (Silva and Perera, 2023). Trees are good candidates for odd prime labeling because they are simply connected graphs that are acyclic. The labeling in this instance is done to preserve the relatively prime condition, which adds to the tree's elegant and well-organized representation. Even though odd prime labeling is not as widely used as some other kinds of graph labeling, it is still beneficial in some theoretical areas of graph theory (Maged *et al.,* 2020).

Further, the particular graph known as the Mobius ladder graph is a unique topic to be dealt with throughout this research work. Mobius ladder graphs are a fascinating class of graph theory graphs identified by their folded and cyclic structure. An extension of the more well-known ladder graph, the Mobius ladder graph has a literal twist. A ladder that folds back on itself to form a closed loop is one way to conceptualize it. Graph theorists are drawn to this unique topological feature because it presents interesting properties and challenges. A standard ladder graph is used to create the Mobius ladder graph by joining its two end vertices to form a circular configuration. The ladder-like rungs on the final graph are still present, but a closed loop adds another layer of complexity. This closed loop gives the graph a cyclic quality that affects several graph-theoretic characteristics (Redha *et al.*, 2021). The Hamiltonian cycle structure of Mobius ladder graphs is one of their most distinctive features. A closed loop that makes exactly one visit to each vertex is known as a Hamiltonian cycle. The folding property of Mobius ladder graphs guarantees the presence of these cycles. The connectivity and traversal properties of Mobius ladder graphs can be understood through the study of Hamiltonian cycles within them. Moreover, graph coloring problems are frequently studied using Mobius ladder graphs as a play area. Determining chromatic numbers and investigating coloring patterns present intriguing challenges due to the cyclic and folded structure. Scholars investigate issues such as the bare minimum of colors required to color the Mobius ladder graph's vertices so that no two adjacent vertices have the same color. Concerns regarding the planarity of the Mobius ladder also arise from its peculiar topology. The ability to draw a graph in a plane with no edges crossing is known as planarity in graph theory. Scholars could investigate the planar embedding of Mobius ladder graphs and examine how their folding structure shapes planar representations. The Mobius ladder graph can also be used to experiment with labeling schemes, like the "odd prime labeling" described above (Faria *et al.,* 2023).

At this point, the main topic of discussion in this research paper will be a particular category of graphs known as the Mobius ladder graph, the odd prime labeling technique. The Mobius ladder graph's odd prime labeling method is an intriguing topic in graph theory. Upon starting this mathematical voyage, we encounter the Mobius ladder graph, a topologically fascinating object that resembles the Mobius strip and forms a closed loop, representing the beauty of combinatory. There are strict requirements for this labeling scheme: labels on adjacent vertices have to differ by odd prime numbers, and the graph has to cover the whole range of odd numbers. This seemingly straightforward task gives life to the Mobius ladder graph through a complex interplay of mathematical symmetries and relationships (Maged *et al*., 2020). In the Mobius ladder graph, each vertex stands for an important component that is awaiting a numerical designation via the odd prime labeling procedure. As we start this labeling process, we see how vertices are interconnected; each label fits into a bigger mathematical puzzle in which the prime differences between vertices define adjacency relationships. Examine the elegant motion of odd primes decorating the graph's vertices in the Mobius ladder. Adjacent vertices resonate with the harmony of their prime disparities when prime numbers are carefully orchestrated, creating a rhythmic pattern. This numerical symphony produces a conceptual and visual spectacle that emphasizes the Mobius ladder graph's natural order and beauty. Beyond its visual appeal, the odd prime labeling reveals interesting characteristics incorporated into the Mobius ladder. The particular prime number configuration at the vertices yields numerous structural insights. Patterns appear as we move up the labeled Mobius ladder, demonstrating the symmetric equilibrium that results from adhering to prime differences. Moreover, the peculiar prime labeling of the Mobius ladder carries over into the domain of exploration and connectivity. Each vertex has a distinct numerical signature that directs connectivity and shapes the relationships that emerge within the graph. The odd prime numbers function as guiding stars, defining the paths and connections between vertices on the labeled Mobius ladder, transforming it into a mathematical landscape that demands study and analysis (Meena and Gajalakshmi, 2022). The prime number distribution plays an important role in the quest for odd prime labeling on the Mobius ladder. The exploration is made more complex and richer by the need to cover all odd primes during the labeling process. This condition turns the labeling task into an all-encompassing numerical puzzle in which every prime number has a specific place and contributes to the overall odd prime labeling of the Mobius ladder. It becomes clear as we go deeper into the odd prime labeling of Mobius ladder graphs that this idea is more than just a mathematical exercise; rather, it opens up new avenues for understanding the beauty and complexity that are intrinsic to graph theory (Maged *et al.,* 2020). On odd prime labeling of graphs. However, the main questions of this research are whether odd prime labeling can be used to label all Mobius ladder graphs and whether a general procedure can be developed to label all Mobius graphs using this specific kind of labeling technique. Therefore, the primary contributions of this research work are the development of a generalized process for determining that all Mobius ladder graphs are odd prime graphs and a general type of proof demonstrating that all Mobius ladder graphs are odd prime.

The list of definitions below is really helpful in conducting further research.

Definition (Graph Labeling)

An assignment of integers to the vertices or edges of a graph, subject to certain conditions, is known as graph labeling.

Definition (Mobius Ladder graph-M_n)

A Mobius ladder, sometimes called a Mobius wheel, of order n is a simple graph obtained by introducing a twist in a prism graph of order n.

Definition (Odd Prime Labeling)

Let $G(V, E)$ be a graph. A bijection $f: V \rightarrow O|v|$ is called an odd prime labeling if for each $uv \in E$, gcd $gcd\ (f(u), f(v)\ ) = 1$. The graph that admits *odd prime labeling* is called an *Odd Prime Graph*.

2. Methodology

2.1. Observation 1:

A series of numbers where every single number is odd is known as an odd sequence. If an integer cannot be divided by two, it is considered an odd number. In the case of the odd sequence$\{1, 3, 5, ..., 2n - 1\}$ (where n is a natural number) can be used to label the vertices of the graphs in the method of odd prime labeling. It is constructed as two distinct odd sequences with the common difference (d) to label all Mobius ladder graphs. To label the all-Mobius ladder graphs in this case, we use two odd sequences with $d = 4$ that start with odd numbers 1 and 3. It is feasible to assign the odd sequences $\{1, 5, 9, 13, 17, ..., 4n - 3\}$ to the outer vertices and $\{3, 7, 11, 15, ..., 4n - 1\}$ to the inner vertices of Mobius ladder graphs. The labeled n^{th} outer vertex can be obtained $O_n = 1 + (n - 1)4 = 4n - 3$ and the labeled n^{th} inner vertex can be obtained $I_n = 3 + (n - 1)4 = 4n - 1$, respectively.

2.2. Observation 2:

Properties of gcd:

For any non-zero integers a, b, c, and any positive integer h, the following hold:

I. $gcd(a, a + 2h) = 1$: if is odd

II. $gcd(a, b) = gcd(a, b - a) = gcd(b, a - b)$

III. $gcd(a, b) = gcd(a + cb, b)$

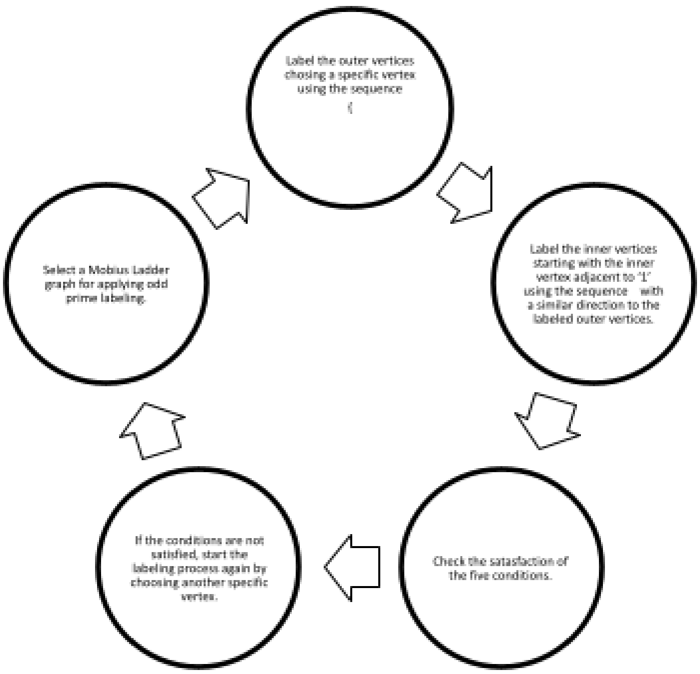

Figure 1: Procedure of labeling the Mobius ladder Graphs

Theorem: *Every Mobius ladder* (M_n) *graph is an odd prime labeling graph.*

Proof:

Consider the following three odd numbers of arithmetic sequences:

(1) $4, 7, 10, 13, 16, ..., 3k + 1$; (2) $3, 6, 9, 12, 15, ..., 3k$; & (3) $5, 8, 11, 14, 17, ..., 3k + 2$ (where k is a natural number). These sequences represent all Mobius ladder graphs (M_n) (where n is the pair of vertices). It now had to take into consideration the five conditions that had to be satisfied for each of the three arithmetic sequences for it to be a Mobius ladder graph (M_n), odd prime.

Condition 01:

Pairwise outer vertices are relatively prime to each other.

That is $gcd(outer\ vertex, outer\ vertex) = 1$

Condition 02:

Pairwise inner vertices are relatively prime to each other.

That is $gcd(inner\ vertex, inner\ vertex) = 1$

Condition 03:

Pairwise outer & inner vertices should be relatively prime to each other.

That is $gcd(outer\ vertex, inner\ vertex) = 1$

Condition 04:

The twisted adjacent vertices, which are the labeled outer vertex 1 and the labeled n^{th} inner vertex or outer n^{th} vertex and the inner $(n - 1)^{th}$, should be relatively prime. (Select a method based on the labeled graph.)

That is $gcd(1, n^{th} inner\ vertex) = 1$

Condition 05:

The twisted adjacent vertices, which are the labeled inner vertex 3 & the labeled outer n^{th} vertex or inner n^{th} and the outer $(n - 1)^{th}$ should be relatively prime. (Select a method based on the labeled graph.)

That is $gcd(3,\ n^{th} outer\ vertex\ ) = 1$

- Now first consider the arithmetic series of $4, 7, 10, 13, 16, ..., 3k + 1$ & each of the below conditions.

Condition 1:

The outer vertex pair $(1, 5), (5, 9), (9, 13), ..., (n - 1, n)$ are adjacent to each other.

Consider the general term $\left[(n - 1)^{th}, n^{th}\right]$

Label the vertex
$T_{n-1} = 1 + (n - 1 - 1)4 = 4n - 7 = 4(3k + 1) - 7 = 12k - 3$

Label the vertex
$T_{n} = 1 + (n - 1)4 = 4n - 3 = 4(3k + 1) - 3 = 12k + 1$

Then,

$gcd\left((n - 1)^{th}, n^{th}\right) = gcd(12k - 3, 12k + 1) = gcd(12k + 1, - 4) = gcd(1, - 4) = 1$

It means that $(n - 1)^{th}$ & n^{th} vertices are relatively prime.

Therefore, all the pair of $(1, 5), (5, 9), (9, 13), ..., (n - 1, n)$ are also relatively prime.

Condition 2:

The inner vertex pair $(3, 7), (7, 11), (11, 15), ..., (n - 1, n)$ are adjacent to each other.

Consider the general term $\left[(n - 1)^{th}, n^{th}\right]$

Label the vertex
$T_{n-1} = 3 + (n - 1 - 1)4 = 4n - 5 = 4(3k + 1) - 5 = 12k - 1$

Label the vertex $T_{n} = 3 + (n - 1)4 = 4n - 1 = 4(3k + 1) - 1 = 12k + 3$

Then,

$gcd\left((n-1)^{th}, n^{th}\right) = gcd(12k-3, 12k+1) = gcd(12k+1, -4) = gcd(1, -4) = 1$

It means that $(n-1)^{th}$ & n^{th} vertices are relatively prime.

Therefore, all the pair of $(3, 7)$, $(7, 11)$, $(11, 15)$, ..., $(n-1, n)$ are adjacent to each other.

Condition 3:

The vertex pair of outer & inner $(1, 3)$, $(5, 7)$, $(9, 11)$, ..., $(n, n+2)$ are also adjacent pair wisely.

Consider the general pair of n^{th} outer vertex & n^{th} inner vertex which can be labeled as follows,

Label the outer vertex

$T_n = 1 + (n-1)4 = 4n - 3 = 4(3k+1) - 3 = 12k + 1$

Label the inner vertex

$T_n = 3 + (n-1)4 = 4n - 1 = 4(3k+1) - 1 = 12k + 3$

Then,

$gcd\left(n^{th}, n^{th}\right) = gcd(12k+1, 12k+3) = gcd(12k+3, -2) = gcd(3, -2) = 1$

It means that the n^{th} outer vertex & n^{th} inner vertex are relatively prime.

Therefore, all the vertex pair of outer & inner $(1, 3)$, $(5, 7)$, $(9, 11)$, ..., $(n, n+2)$ are relatively prime.

Condition 4:

Consider the twisted adjacent vertices, which are the outer vertex 1 and the n^{th} inner vertex.

Label the n^{th} inner vertex

$T_n = 3 + (n-1)4 = 4n - 1 = 4(3k+1) - 1 = 12k + 3$

Then, $gcd(1, n^{th}) = gcd(1, 12k + 3) = 1$

Therefore, labeled outer vertex 1 and the n^{th} inner vertex are relatively prime to each other.

Condition 5:

Consider the twisted adjacent vertices, which are the inner vertex 3 & the n^{th} outer vertex.

Label the n^{th} outer vertex
$T_n = 1 + (n - 1)4 = 4n - 3 = 4(3k + 1) - 3 = 12k + 1$

Then, $gcd(3, n^{th}) = gcd(3, 12k + 1) = gcd(3, 1) = 1$

Therefore, labeled inner vertex 3 and the n^{th} outer vertex are relatively prime to each other.

- ❖ Now consider the next arithmetic series $3, 6, 9, 12, 15, ..., 3k$ & each of the below conditions.

Condition 1:

The outer vertex pair $(1, 5), (5, 9), (9, 13), ..., (n - 1, n)$ are adjacent to each other.

Consider the general term $\left[(n - 1)^{th}, n^{th}\right]$

Label the vertex
$T_{n-1} = 1 + (n - 1 - 1)4 = 4n - 7 = 4(3k) - 7 = 12k - 7$

Label the vertex $T_n = 1 + (n - 1)4 = 4n - 3 = 4(3k) - 3 = 12k - 3$

Then,

$gcd\left((n - 1)^{th}, n^{th}\right) = gcd(12k - 7, 12k - 3) = gcd(12k - 3, - 4) = gcd(- 3, - 4) = 1$

It means that $(n - 1)^{th}$ & n^{th} outer vertices are relatively prime.

Therefore, all the pair of $(1, 5), (5, 9), (9, 13), ..., (n - 1, n)$ are also relatively prime.

Condition 2:

The inner vertex pair $(3, 7), (7, 11), (11, 15), ..., (n - 1, n)$ are adjacent to each other.

Consider the general term $\left[(n - 1)^{th}, n^{th}\right]$

Label the vertex
$T_{n-1} = 3 + (n - 1 - 1)4 = 4n - 5 = 4(3k) - 5 = 12k - 5$

Label the vertex $T_{n} = 3 + (n - 1)4 = 4n - 1 = 4(3k) - 1 = 12k - 1$

Then,
$gcd\left((n - 1)^{th}, n^{th}\right) = gcd(12k - 5, 12k - 1) = gcd(12k - 5, 4) = gcd(- 5, 4) = 1$

It means that $(n - 1)^{th}$ & n^{th} inner vertices are relatively prime.

Therefore, all the pair of $(3, 7), (7, 11), (11, 15), ..., (n - 1, n)$ are adjacent to each other.

Condition 3:

The vertex pair of outer & inner $(1, 3), (5, 7), (9, 11), ..., (n, n + 2)$ are also adjacent pairs wisely.

Consider the general pair of n^{th} outer vertex & n^{th} inner vertex which can be labeled as follows,

Label the outer vertex
$T_{n} = 1 + (n - 1)4 = 4n - 3 = 4(3k) - 3 = 12k - 3$

Label the inner vertex $T_{n} = 3 + (n - 1)4 = 4n - 1 = 4(3k) - 1 = 12k - 1$

Then,
$gcd(n^{th}, n^{th}) = gcd(12k - 3, 12k - 1) = gcd(12k - 3, 2)gcd(- 3, 2,) = 1$

It means that the n^{th} outer vertex & n^{th} inner vertexes are relatively prime.

Therefore, all the vertex pair of outer & inner $(1, 3), (5, 7), (9, 11), ..., (n, n + 2)$ are relatively prime.

Condition 4:

Consider the twisted adjacent vertices, which are the $(n-1)^{th}$ outer vertex & the n^{th} inner vertex.

Label $(n-1)^{th}$ outer vertex

$$T_{n-1} = 1 + (n-1-1)4 = 4n - 7 = 4(3k) - 7 = 12k - 7$$

Label the n^{th} inner vertex

$$T_n = 3 + (n-1)4 = 4n - 1 = 4(3k) - 1 = 12k - 1$$

Then,

$$gcd\left((n-1)^{th}, n^{th}\right) = gcd(12k-7, 12k-1) = gcd(12k-7, 6) = gcd(-7, 6) = 1$$

Therefore, the $(n-1)^{th}$ outer vertex & the n^{th} inner vertex are relatively prime to each other.

Condition 5:

Consider the twisted adjacent vertices, which are the n^{th} outer vertex & the $(n-1)^{th}$ inner vertex.

Label the n^{th} outer vertex

$$T_n = 1 + (n-1)4 = 4n - 3 = 4(3k) - 3 = 12k - 3$$

Label the $(n-1)^{th}$ inner vertex

$$T_{n-1} = 3 + (n-1-1)4 = 4n - 5 = 4(3k) - 5 = 12k - 5$$

Then,

$$gcd\left(n^{th}, (n-1)^{th}\right) = gcd(12k-3, 12k-5) = gcd(12k-3, -2) = gcd(-3, -2) = 1$$

Therefore, the n^{th} $outer\ vertex$ & $(n-1)^{th}$ inner vertexes are relatively prime.

- ❖ Now consider the last arithmetic series $5, 8, 11, 14, 17, \ldots, 3k+2$ & each of the below conditions.

Condition 1:

The outer vertex pair $(1, 5), (5, 9), (9, 13), ..., (n - 1, n)$ are adjacent to each other.

Consider the general term $\left[(n - 1)^{th}, n^{th}\right]$

Label the vertex $T_{n-1} = 1 + (n - 1 - 1)4 = 4n - 7 = 4(3k + 2) - 7 = 12k + 1$

Label the vertex $T_{n} = 1 + (n - 1)4 = 4n - 3 = 4(3k + 2) - 3 = 12k + 5$

Then, $gcd\left((n - 1)^{th}, n^{th}\right) = gcd(12k + 1, 12k + 5) = gcd(12k + 1, 4) = gcd(1, 4)$ =1

It means that $(n - 1)^{th}$ & n^{th} outer vertices are relatively prime.

Therefore, all the pairs of $(1, 5), (5, 9), (9, 13), ..., (n - 1, n)$ are also relatively prime to each other.

Condition 2:

The inner vertex pair $(3, 7), (7, 11), (11, 15), ..., (n - 1, n)$ are adjacent to each other.

Consider the general term $\left[(n - 1)^{th}, n^{th}\right]$

Label the vertex
$T_{n-1} = 3 + (n - 1 - 1)4 = 4n - 5 = 4(3k + 2) - 5 = 12k + 3$

Label the vertex $T_{n} = 3 + (n - 1)4 = 4n - 1 = 4(3k + 2) - 1 = 12k + 7$

Then, $gcd\left((n - 1)^{th}, n^{th}\right) = gcd(12k + 3, 12k + 7) = gcd(12k + 3, 4) = gcd(3, 4) =$
1

Therefore, all the pair of $(3, 7), (7, 11), (11, 15), ..., (n - 1, n)$ are adjacent to each other.

Condition 3:

The vertex pair of outer & inner $(1, 3), (5, 7), (9, 11), ..., (n, n + 2)$ are also adjacent pairs wisely.

Consider the general pair of n^{th} outer vertex & n^{th} inner vertex can be labeled as follows,

Label the outer vertex
$T_n = 1 + (n - 1)4 = 4n - 3 = 4(3k + 2) - 3 = 12k + 5$

Label the inner vertex

$T_n = 3 + (n - 1)4 = 4n - 1 = 4(3k + 2) - 1 = 12k + 7$

Then,
$gcd(n^{th}, n^{th}) = gcd(12k + 5, 12k + 7) = gcd(12k + 5, 2)gcd(5, 2,) = 1$

It means that the n^{th} outer vertex & n^{th} inner vertexes are relatively prime.

Therefore, all the vertex pair of outer & inner $(1, 3), (5, 7), (9, 11), s..., (n, n + 2)$ are relatively prime.

Condition 4:

Consider the twisted adjacent vertices which are $(n - 1)^{th}$ $outer\ vertex$ & n^{th} inner vertex.

Label the $(n - 1)^{th}$ outer
$T_{n-1} = 1 + (n - 1 - 1)4 = 4n - 7 = 4(3k + 2) - 7 = 12k + 1$

Label the n^{th} inner vertex

$T_n = 3 + (n - 1)4 = 4n - 1 = 4(3k + 2) - 1 = 12k + 7$

Then,
$gcd\left((n - 1)^{th}, n^{th}\right) = gcd(12k + 1, 12k + 7) = gcd(12k + 1, 6) = gcd(1, 6) = 1$

Therefore, the outer vertex $(n - 1)^{th}$ & n^{th} inner vertex are relatively prime to each other.

Condition 5:

Consider the twisted adjacent vertices, which are the n^{th} outer vertex & the $(n-1)^{th}$ inner vertex.

Label the n^{th} outer vertex $T_n = 4n - 3 = 4(3k + 2) - 3 = 12k + 5$

Label the $(n-1)^{th}$ inner vertex

$$T_{n-1} = 3 + (n - 1 - 1)4 = 4n - 5 = 4(3k + 2) - 5 = 12k + 3$$

Then,

$$gcd\left(n^{th}, (n-1)^{th}\right) = gcd(12k + 512k + 3) = gcd(12k + 3, 2) = gcd(3, 2) = 1$$

Therefore, the n^{th} outer vertex & $(n-1)^{th}$ inner vertex are relatively prime to each other.

Hence, all the conditions are satisfied for every arithmetic sequence, which means that *every Mobius ladder* (M_n) *graph is an odd prime labeling graph*.

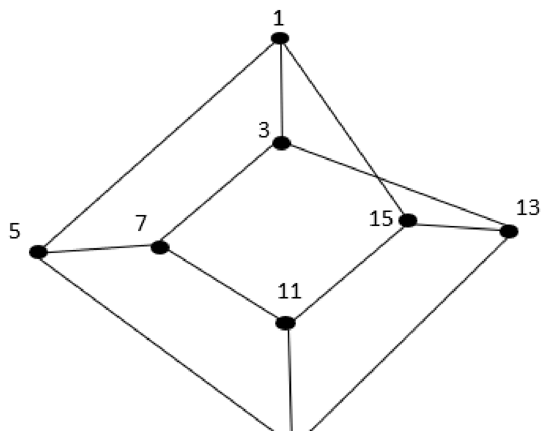

Figure 1: Mobius ladder graph (M_4)

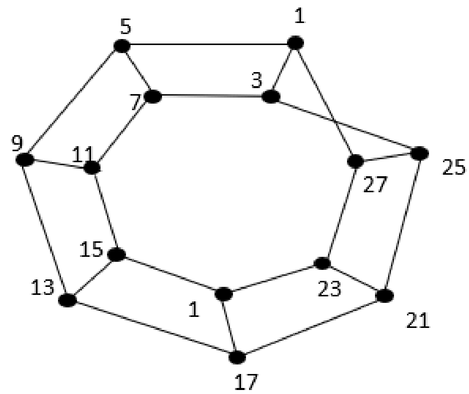

Figure 2: Mobius ladder graph (M_7)

Now, consider how odd prime labeling can be used to label Figure 1 of the Mobius ladder graph.

The outer vertices of the Mobius Ladder graph can be labeled using the odd sequence $\{1, 5, 9, 13, 17, ..., 4n - 3\}$ first. This can be done by choosing a specific vertex (one with twisted edges) and labeling it as 1 in an anticlockwise direction up to an odd number 13, which is then assigned to the other twisted vertex. Next, the inner vertices can be labeled using the odd sequence $\{3, 7, 11, 15, \dots, 4n - 1\}$ starting with the odd number 3, the vertex adjacent to 1 is labeled in the same direction as previously mentioned, up to the odd number 15.

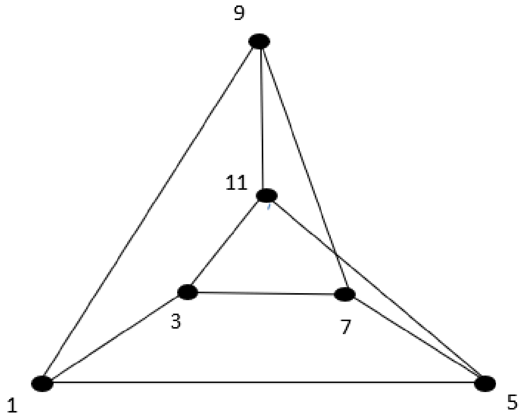

Figure 3: Mobius ladder graph (M_3)

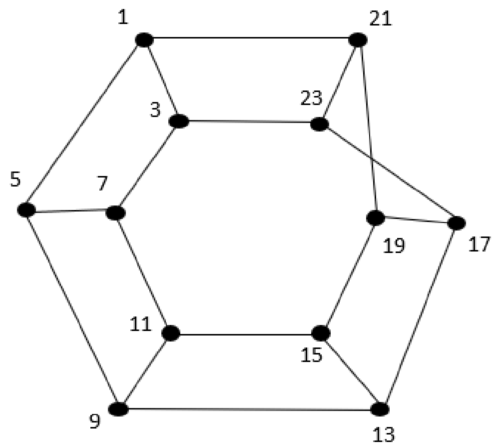

Figure 4: Mobius ladder graph (M_6)

First, the outer vertices can be labeled using the odd sequence $\{1, 5, 9, 13, 17, \dots, 4n - 3\}$ by choosing a particular vertex (closest vertex to the twisted-edged vertex adjacent to it) and labeling it as 1 in an anticlockwise direction up to odd number 21, which is then assigned to the twisted vertex. Next, the inner vertices can be labeled using the odd sequence $\{3, 7, 11, 15, \dots, 4n - 1\}$ by starting with the odd number 3, the vertex adjacent to 1 is labeled in the same direction as previously mentioned, up to the odd number 23.

3. Results & Discussion

One of the most intriguing and recently studied areas is odd prime labeling, which is defined as labeling the vertices with the set of all odd integers, where the two adjacent vertices are relatively prime. The Mobius ladder graphs have been labeled by introducing a general proof that all Mobius ladder graphs are odd primes. Here, it is shown that, under five special conditions, all Mobius ladder graphs are odd prime labeling graphs. The most important thing was to use only odd numbers to identify every single vertex. In addition, the proof could be built using three distinct arithmetic series under specific circumstances; thus, when all the series were added together, they represented each Mobius ladder graph. In this case, however, the gcd equals 1, and each condition was centered at the relative prime property satisfaction. As a result, this research has led to the introduction of a new labeling type for Mobius ladder graphs, which we refer to in this paper as an odd prime labeling of Mobius ladder graphs. This is accomplished by employing two distinct kinds of odd sequences and by adhering to the general definition of odd prime labeling. In addition, innovative mathematical solutions are required for many real-world problems, and graph labeling techniques can be effectively applied to address some of these challenges. The study presents encouraging prospects for investigating these kinds of applications. Because of its special characteristics, this labeling technique may help develop useful solutions for domains including security systems, network design, and cryptography. Mobius ladder graphs' unique labeling patterns, for example, could be utilized in cryptography to develop strong data encryption techniques. Similarly, by improving security mechanisms and boosting connectivity, odd prime labeling may help network designers create safe and effective networks. This research paves the way for significant practical applications that could enhance communication infrastructure and digital security by examining the characteristics of odd prime labeling.

4. Conclusion

This novel odd prime labeling technique has enabled a deeper understanding of the odd prime labeling of Mobius ladder graphs and the collection of important new information about their structural characteristics. The unique structure of Mobius ladder graphs, coupled with the constraints imposed by odd prime labeling, provides a rich landscape for exploration and discovery in the realm of mathematics. Every vertex on the Mobius ladder graph is labeled with a unique odd number in this labeling scheme. The improvement starts with labels applied to adjacent vertices that differ exactly by a prime number. The odd prime labeling rules, which require adjacent vertices to differ by a prime number, result in a harmonious dance of numbers across the graph. The significance of odd primes adds a mathematical layer of elegance, highlighting their exclusivity and indivisibility. As we navigate this numerical landscape, patterns appear, highlighting the profound relationship between prime numbers and the structural intricacies of Mobius ladder graphs. This research not only reveals the visual appeal of mathematical symmetries but also advances our understanding of the interconnectedness of prime numbers within graph structures.

When considering the methodology presented in this paper, it is constructed in a unique manner, making it simple to grasp the idea and scenario within this labeling method, which is extensively explained for the reader's purposes and interests. Furthermore, the latest method for finding odd prime labeling can be used to label Mobius ladder graphs and takes into account two odd sequences. This can be demonstrated through a general proof that considers the five necessary conditions, which are also illustrated in detail in this paper, and apply to three arithmetic series. Therefore, the novel approach introduced in this paper leads to the conclusion that odd prime labeling can be used to label any Mobius ladder graph under some conditions. In addition, prime labeling in the Mobius ladder graph consequently guarantees promising potentiality in cryptographic applications. The investigation may reveal, in a future research direction, new ways of exploiting the unique structural properties of the Mobius ladder graph in enhancing cryptographic systems. The extension of this work may explore whether odd prime labeling remains valid as outer rungs are added to such graphs, thereby showing new layers of mathematical complexity and security. Another avenue that could be worthy of research would be the study of the union of star graphs and Mobius ladder graphs with odd prime labeling, which might offer new insight into how diverse graph structures can be combined to develop cryptographic techniques at a high level. Beyond deepening the understanding of odd prime labeling, these avenues also open a door to fresh applications in data encryption and secure communications.

5. References

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