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Research Article

Calibrating Life Annuities for Life Insurance and Pension Fund Implementations: A Mathematical Model

G.M. Ogungbenle[1*], A.B. Sogunro[2] and O.E. Ogungbenle[3]

**moyosiolorun@gmail.com*

[1]*Department of Actuarial Science, Faculty of Management Sciences, University of Jos, Nigeria*

[2] *Department of Actuarial Science and Insurance, Faculty of Management Sciences University of Lagos, Nigeria*

[3]*Department of Physiology, Faculty of Natural Sciences, University of Ilorin, Nigeria*

Abstract

*Annuity schemes are developed to provide different financial retirement instruments to address the problems of income of retirees. Annuities schemes provide additional financial retirement planning option with minimal credit risk and a certain level of liquidity to the retirees by turning a lump-sum premiums into life time streams of monthly income at a reasonable and a stable rate. In this study we establish a mathematical framework to construct an actuarial model that enables us to estimate life annuity schemes. In order to help actuaries and life office in the administration of pension schemes, the objectives of this study are anchored on the followings to **(i)** construct numerical estimations of life annuity through single life parameterization of Makeham's law* $GM(1,2)$ *through algebraic technique **(ii)** apply the mean value theorem to construct modification theorems for* $GM(1,2)$ *under the framework of policy alterations **(iii)** apply the commutation function to develop mathematical model for the actuarial cash inflows and outflows. It is observed from the computational results that the trend of annuities progressively reduces downwards from ages* $30-90$. *This trend justifies the mathematical behavior that the continuous annuity integral* $\left(\bar{a}_x\right)$ *is a decreasing function. To demonstrate how long the annuitants are covered by the decreasing life insurance scheme, our results presents the terms of the deceasing whole life annuity scheme across the ages. The older annuitants will pay a higher percentage of premium for the whole life insurance scheme and consequently can only enjoy a shorter term of insurance coverage since at senescence, annuitants earn higher payments and have a higher mortality rate. The smaller the interest rate becomes, the bigger the proportion the of the term* *insurance and smaller costs for the life annuity hence proportions of the mortality component are usually higher for the old annuitants.*

Keywords: *Annuitants, commutation function, gamma functions, parameterization, whole life annuity*

1. Introduction

1.1. Introduction to GM (m,n) class

Many employees prefer life annuity option at retirement during which their accrued savings are transferred to a life underwriting life office. In fact most life tables which are generated from functionally age dependent mortality laws are Makehamised. Age dependent mortality rates are often defined to compute life annuities used in pensions management. For example, in Kwong, Chan and Siu-Hang Li (2019), a pension fund scheme is developed for use in Singapore that needs the application of life annuity as developed in this paper. When the continuous parsimonious parametric mortality intensities are Makehamised, then the life table functions used in computing the actuarial present values of the fully continuous whole life annuity could be expressed in terms of special functions such as Gamma, incomplete lower Gamma and incomplete upper Gamma functions for a homogeneous insured population. In this paper, the objectives are to (i) construct numerical estimations of life annuity through single life parameterization of Makeham's law $GM(1,2)$ through algebraic technique (ii) apply the mean value theorem to construct modification theorems for $GM(1,2)$ under the framework of policy alterations (iii) apply the commutation function to develop mathematical model for the actuarial cash inflows and outflows In human mortality, the $GM(1,2)$ intensity is applied to define the trends of mortality where the management of life office assets and liability depend on the death rate of the insured (Siswono, Azmi, and Syaifudin 2021). Where $GM(m,n) = \sum_{k=1}^{m} \beta_k x^{k-1} + \exp \sum_{k=m+1}^{m+n} \beta_k x^{k-m-1}; m+n<5$. Following Lageras, (2009) and Missov and Lennart, (2013), continuous parametric functions such as $GM(0,2)$ assumes that mortality rate increases as age advances. However, the $GM(1,2)$ intensity law adds an age independent parameter which is not associated with senescence. In human populations, issues connected with overestimation in observed death rates at senescence in $GM(0,2)$ mortality trajectories aroused the study of continuous parsimonious parametric mortality models which are responsible for the unobserved heterogeneity and consequently, the cohort population is then partitioned into strata in accordance with an observed measure of insured's exposure to the risk of death. In Dragan, (2022), we have observed that the methods of generating mortality tables was initially developed for cohorts whose members have varying characteristics in connection with longevity measures. In life insurance underwriting, the goal of mortality computations is to assess the financial effect of death or survivorship. In order to compute the evolving payout on a life insurance policy or annuity, the sum at risk would be multiplied by the mortality rate of an insured while the benefit amount should be multiplied by the survival probability of the annuitant. As a result, there exists a correlation between the benefit amount or the sum insured and the mortality rates. Apparently, mortality is reduced for the insured with the bigger of these amounts and this skewness in the mortality by amounts constitutes direct financial consequences.

1.2. Numerical Estimation of the $GM(1,2)$ Parameters

Following Neil (1979); Debon, Montes and Sala (2005); Chowdhury (2012); Kara (2021); Patricio, Castellares and Queiroz (2023), the continuous $GM(1,2)$ is defined as

$$\mu_x = \rho + GH^x \tag{1}$$

Let $\zeta = e^{\rho}$ and $G = -\log_e \delta \log_e H$, $\zeta > 0$ and $\delta > 0$

The right hand side must be multiplied by (-1) throughout by definition of the force of mortality

$$\mu_x = -\log_e \zeta + (-\log_e \delta \log_e H) H^x \tag{2}$$

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{d_x} = -\frac{d\log_e l_x}{dx} \tag{3}$$

$$\mu_x = -\frac{d\log_e l_x}{dx} = -\log_e \zeta + (-\log_e \delta \log_e H) H^x \tag{4}$$

Taking K as the constant of integration,

$$-\int \frac{d\log_e l_x}{dx}dx = \int -\log_e\zeta + \left(-\log_e \delta \log_e H\right) H^x dx + K \tag{5}$$

$$\log_e l_x = x\log_e \zeta + \left(\log_e \delta \log_e H\right)\frac{H^x}{\log_e H} + \log_e \lambda \tag{6}$$

$$K = \log_e \lambda$$

$$\log_e l_x = \log_e \zeta^x + \left(\log_e \delta \log_e H\right)\frac{H^x}{\log_e H} + \log_e \lambda \tag{7}$$

$$\log_e l_x = \log_e \zeta^x + \left(\log_e \delta\right) H^x + \log_e \lambda \tag{8}$$

where $\log_e \lambda$, is the constant of integration.

$$\log_e l_x = \log_e \zeta^x + \left(\log_e \delta^{H^X}\right) + \log_e \lambda = \log_e \lambda\zeta^x\delta^{H^X} \tag{9}$$

Now, equating both sides, we have

$$l_x = \lambda\zeta^x\delta^{H^x} \Rightarrow \int_0^\infty l_{x+s}\mu_{x+s}ds = \lambda\zeta^x\delta^{H^X} \tag{10}$$

Note that the age of the insured is chronological. We can take four of such age with equal intervals at the points $\{x+0, x+s, x+2s, x+3s\}$ to have four systems of simultaneous equations

$$l_{x+s} = \lambda\zeta^{x+s}\delta^{H^{x+s}} \tag{11}$$

$$l_{x+2s} = \lambda\zeta^{x+2s}\delta^{H^{x+2s}} \tag{12}$$

$$l_{x+3s} = \lambda\zeta^{x+3s}\delta^{H^{x+3s}} \tag{13}$$

$${}_sP_x = \frac{l_{x+s}}{l_x} = \frac{\lambda\zeta^{x+s}\delta^{H^{x+s}}}{\lambda\zeta^x\delta^{H^x}} = \frac{\zeta^s\delta^{H^{x+s}}}{\delta^{H^x}} = \zeta^s\delta^{H^{x\left(H^S-1\right)}} \tag{14}$$

Considering 4 consecutive values of function $\log_e l_x$

$$\log_e l_{x+0} = x\log_e \zeta + \left(\log_e \delta\right)H^x + \log_e \lambda \tag{15}$$

$$\log_e l_{x+s} = (x+s)\log_e \zeta + (\log_e \delta)H^{x+s} + \log_e \lambda \tag{16}$$

$$\log_e l_{x+2s} = (x+2s)\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda \tag{17}$$

$$\log_e l_{x+3s} = (x+3s)\log_e \zeta + (\log_e \delta)H^{x+3s} + \log_e \lambda \tag{18}$$

$$\begin{aligned}&\log_e l_{x+s} - \log_e l_x = (x+s)\log_e \zeta + (\log_e \delta)H^{x+s} + \log_e \lambda \\ &-x\log_e \zeta + (\log_e \delta)H^x + \log_e \lambda\end{aligned} \tag{19}$$

$$\log_e l_{x+s} - \log_e l_x = (s\log_e \zeta) + (\log_e \delta)H^s H^x - (\log_e \delta)H^x \tag{20}$$

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + H^x(H^s - 1)\log_e \delta \tag{21}$$

$$\log_e l_{x+2s} - \log_e l_{x+s} = s\log_e \zeta + H^{x+s}(H^s - 1)\log_e \delta \tag{22}$$

$$\log_e l_{x+3s} - \log_e l_{x+2s} = s\log_e \zeta + H^{x+2s}(H^s - 1)\log_e \delta \tag{23}$$

$$\begin{aligned}&\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (x+2s)\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda - \\ &2\left[(x+s)(\log_e \zeta) + (\log_e \delta)H^{x+s} + \log_e \lambda\right] + x(\log_e \zeta) + (\log_e \delta)H^x + (\log_e \lambda)\end{aligned} \tag{24}$$

$$\begin{aligned}&\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = x\log_e \zeta + 2s\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda \\ &-2x\log_e \zeta - 2s\log_e \zeta - 2(\log_e \delta)H^{x+s} - 2\log_e \lambda + x\log_e \zeta + (\log_e \delta)H^x + \log_e \lambda\end{aligned} \tag{25}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)H^{x+2s} - 2(\log_e \delta)H^{x+s} + (\log_e \delta)H^x \tag{26}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)H^x\left[H^{2s} - 2H^s + 1\right] \tag{27}$$

Let $U = H^s$, then

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)U\left[U^2 - 2U^s + 1\right] \tag{28}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)U(U-1)^2 \tag{29}$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)H^x(H^x-1)^2 \tag{30}$$

Similarly,

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (x+3s)\log_e \zeta + (\log_e \delta) H^{x+3s} + \log_e \lambda$$
$$-2\left[(x+2s)\log_e \zeta + (\log_e \delta) H^{x+2s} + \log_e \lambda\right] + (x+s)\log_e \zeta + (\log_e \delta) H^{x+s} \quad (31)$$
$$+\log_e \lambda$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = x\log_e \zeta + 3s\log_e \zeta + (\log_e \delta) H^{x+3s} + \log_e \lambda$$
$$-2x\log_e \zeta - 4s\log_e \zeta - 2(\log_e \delta) H^{x+2s} - 2\log_e \lambda + x\log_e \zeta + s\log_e \zeta \quad (32)$$
$$+(\log_e \delta) H^{x+s} + \log_e \lambda$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (\log_e \delta) H^{x+3s} - 2(\log_e \delta) H^{x+2s} + (\log_e \delta) H^{x+s}$$
(33)

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (\log_e \delta) H^{x+s}\left[H^{2s} - 2H^s + 1\right] \quad (34)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = H^{x+s}\left(H^s - 1\right)^2 \log_e \delta \quad (35)$$

$$\frac{\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s}}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} = \frac{= H^{x+s}\left(H^s - 1\right)^2 \log_e \delta}{H^x\left(H^s - 1\right)^2 \log_e \delta} = H^x \quad (36)$$

Since l_x values are obtained from the continuous registration system, then let

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = \alpha \quad (37)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = \beta \quad (38)$$

$$H^x\left(H^s - 1\right)^2 \log_e \delta = \alpha \quad (39)$$

$$H^{x+s}\left(H^s - 1\right)^2 \log_e \delta = \beta \quad (40)$$

Taking logarithms of the two equations above, we have

$$x\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \quad (41)$$

$$(x+s)\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \beta \quad (42)$$

Subtracting equation (41) from (42)**,** we obtain

$$x\log_e H + s\log_e H + 2\log\left(H^s - 1\right) + \log_e \log_e \delta - x\log_e H - 2\log_e\left(H^s - 1\right) \\ -\log_e \log_e \delta = \log_e \beta - \log_e \alpha \tag{43}$$

$$s\log_e H = \log_e \beta - \log_e \alpha \tag{44}$$

$$\log_e H = \frac{\log_e \beta - \log_e \alpha}{s} = \frac{\log_e \frac{\beta}{\alpha}}{s} \tag{45}$$

$$x\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \tag{46}$$

substituting (45) in (41), we have

$$\frac{x}{s}\log_e \frac{\beta}{\alpha} + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \tag{47}$$

$$\log_e \log_e \delta = \log_e \alpha - \frac{x}{s}\log_e \frac{\beta}{\alpha} - 2\log_e\left(H^s - 1\right) \tag{48}$$

$$\log_e\left[\log_e \delta\right] = \log_e \alpha + \log_e\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}} + \log_e\left(H^s - 1\right)^{-2} = \log_e\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}}\left(H^s - 1\right)^{-2} \tag{49}$$

Equation (49) then becomes

$$\log_e \delta = \alpha\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}}\left(H^s - 1\right)^{-2} \tag{50}$$

$$x\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \tag{51}$$

is re-expressed as

$$x\log_e H + \log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha - \log_e\left(H^s - 1\right) \tag{52}$$

$$\log_e\left[H^x\left(H^s - 1\right)\log_e \delta\right] = \log_e \frac{\alpha}{\left(H^s - 1\right)} \tag{53}$$

$$H^x\left(H^s - 1\right)\log_e \delta = \frac{\alpha}{\left(H^s - 1\right)} \tag{54}$$

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + H^x\left(H^s - 1\right)\log_e \delta \tag{55}$$

Substituting equation (54) in (55), we have

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + \frac{\alpha}{\left(H^s - 1\right)} \tag{56}$$

$$s\log_e \zeta = \log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)} \tag{57}$$

$$\log_e \zeta = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s}\right] \tag{58}$$

Recall from (52) that

$$x\log_e H + \log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha - \log_e\left(H^s - 1\right) \tag{59}$$

$$x\log_e H + \log_e \log_e \delta = \log_e \alpha + \log_e\left(H^s - 1\right)^{-2} \tag{60}$$

$$\log_e\left[H^x \log_e \delta\right] = \log_e \alpha\left(H^s - 1\right)^{-2} \tag{61}$$

$$H^x \log_e \delta = \alpha\left(H^s - 1\right)^{-2} \tag{62}$$

$$H^x = \frac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2} \tag{63}$$

Recall that $x\log_e \zeta + (\log_e \delta)H^x + \log_e \lambda = \log_e l_x$ (64)

$$\log_e \lambda = \log_e l_x - x\log_e \zeta - (\log \delta)H^x \tag{65}$$

putting (58), (63) into (64)

$$\log_e \lambda = \log_e l_x - x\left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s}\right] - \alpha\left(H^s - 1\right)^{-2} \tag{66}$$

$$\log_e \zeta = \rho$$

$$\rho = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] \tag{67}$$

and by the initial definition $G = -\log_e \delta \log_e H$

$$\text{recall } \log_e H = \frac{\log_e \frac{\beta}{\alpha}}{s} = \log_e \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s}} \tag{68}$$

$$H = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s}} \tag{69}$$

$$\text{Note that } G = \frac{\left(-\log_e \delta\right)}{s} \log_e \frac{\beta}{\alpha} \tag{70}$$

$$\text{And } H^x = \frac{\alpha}{\log_e \delta} \left(H^s - 1\right)^{-2} \tag{71}$$

$\mu_x = \rho + GH^x$ becomes

$$\mu_x = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] + \left[\frac{\left(-\log_e \delta\right)}{s} \log_e \frac{\beta}{\alpha} \right] \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \tag{72}$$

$$\mu_x = \left[\frac{\log_e \frac{\log_e l_{x+s}}{l_x} - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s} \log_e \frac{\beta}{\alpha} \tag{73}$$

$$\mu_x = \left[\frac{\log_e \left({}_sP_x\right) - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s} \log_e \frac{\beta}{\alpha} \tag{74}$$

$$\mu_x = \left[\frac{\log_e \zeta^2 \delta^{H^x(H^s-1)} - \frac{\alpha}{(H^s-1)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s} \log_e \frac{\beta}{\alpha} \quad (75)$$

Recall $H = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s}} \Rightarrow H^s = \left(\frac{\beta}{\alpha}\right)$ (76)

So when $s = x$, we have

$$H^x = \left(\frac{\beta}{\alpha}\right) \quad (77)$$

$$\frac{\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s}}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} = \frac{= H^{x+s}\left(H^s-1\right)^2 \log_e \delta}{H^x\left(H^s-1\right)^2 \log_e \delta} = H^x = \left(\frac{\beta}{\alpha}\right) \quad (78)$$

Hence, we obtain

$$\mu_x = \left[\frac{\log_e \zeta^2 \delta^{H^x(H^s-1)} - \frac{\alpha}{(H^s-1)}}{s} \right] + \quad (79)$$

$$\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s} \log_e \frac{(\log_e l_{x+3s} - 2\log_e l_{x+2s} \log_e l_{x+s})}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x}$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x(H^s-1)} - \frac{\alpha}{(H^s-1)}}{s} \right] +$$

$$\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s} \log_e \frac{\left(\log_e \lambda\zeta^{x+3s}\delta^{H^{x+3s}} - 2\log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}} + \log_e \lambda\zeta^{x+s}\delta^{H^{x+s}}\right)}{\log_e \lambda\zeta^{x+2s}\delta^{H^{x+3s}} - 2\log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} + \log_e \lambda\zeta^{x}\delta^{H^{x}}}$$

(80)

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x \left(H^s - 1\right)} - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s} \log_e \frac{\left(\begin{array}{l} \log_e \lambda\zeta^{x+3s}\delta^{H^{x+3s}} - \log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}} + \log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} \\ -\log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}} \end{array} \right)}{\begin{array}{l} \log_e \lambda\zeta^{x+2s}\delta^{H^{x+3s}} - \log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} + \log_e \lambda\zeta^{x}\delta^{H^{x}} \\ -\log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} \end{array}} \tag{81}$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x \left(H^X - 1\right)} - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s} \log_e \frac{\left(\log_e \frac{\lambda\zeta^{x+3s}\delta^{H^{x+3s}}}{\log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}}} + \log_e \frac{\lambda\zeta^{x+s}\delta^{H^{x+s}}}{\lambda\zeta^{x+2s}\delta^{H^{x+2s}}} \right)}{\log_e \frac{\lambda\zeta^{x+2s}\delta^{H^{x+3s}}}{\log_e \lambda\zeta^{x+s}\delta^{H^{x+s}}} + \log_e \frac{\log_e \lambda\zeta^{x}\delta^{H^{x}}}{\lambda\zeta^{x+s}\delta^{H^{x+s}}}} \tag{82}$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x \left(H^X - 1\right)} - \frac{\alpha}{\left(H^s - 1\right)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{\left(-\log_e \delta\right)}{s} \log_e \left[\frac{\left(\log_e \frac{\delta^{H^{x+3s}}}{\delta^{H^{x+2s}}} \frac{\delta^{H^{x+s}}}{\delta^{H^{x+2s}}} \right)}{\log_e \frac{\delta^{H^{x+3s}}}{\delta^{H^{x+s}}} \frac{\delta^{H^{x}}}{\delta^{H^{x+s}}}} \right] \tag{83}$$

1.3. Modification Theorem

The modification theorem allows us to obtain the probability of death when a policy holder changes from one scheme to another and the life insurer has to modify the conditions and terms of the policy actuarially.

$$e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)}=\frac{\left[4-\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} \tag{84}$$

Proof

In Walters and Wilkie (1987); Castro-Perez, Aguilar-Sanchez and Gonzalez-Nucamendi (2020) for a mortality table, the following relationship holds

$$l_U\mu_U=\frac{[l_x+l_{x+\Delta}]}{2}\frac{(\mu_x+\mu_{x+\Delta})}{2} \tag{85}$$

Multiplying both sides of (85) by Δ

$$\Delta l_U\mu_U=\Delta\frac{l_x(\mu_x+\mu_{x+\Delta})}{4}+\Delta\frac{(\mu_x+\mu_{x+\Delta})l_{x+\Delta}}{4} \tag{86}$$

But by the mean value theorem, $l_x-l_{x+\Delta}=\Delta\times l_U\mu_U$ (87)

Putting (87) in (86), we have

$$l_x-l_{x+\Delta}=\Delta\frac{l_x(\mu_x+\mu_{x+\Delta})}{4}+\Delta\frac{(\mu_x+\mu_{x+\Delta})l_{x+\Delta}}{4} \tag{88}$$

$$l_x-\Delta\frac{l_x(\mu_x+\mu_{x+\Delta})}{4}=l_{x+\Delta}+\Delta\frac{(\mu_x+\mu_{x+\Delta})l_{x+\Delta}}{4} \tag{89}$$

Factorising the $l_{(.)}$ in (89)

$$l_x\left[1-\Delta\frac{(\mu_x+\mu_{x+\Delta})}{4}\right]=l_{x+\Delta}\left[1+\Delta\frac{(\mu_x+\mu_{x+\Delta})}{4}\right] \tag{90}$$

Therefore, the survival probability is $_{\Delta}p_x=\dfrac{\left[1-\Delta\dfrac{(\mu_x+\mu_{x+\Delta})}{4}\right]}{\left[1+\Delta\dfrac{(\mu_x+\mu_{x+\Delta})}{4}\right]}$ (91)

Substituting $\mu_x=GM(1,2)=a+e^be^{Cx}$ in (91)

$$ {}_{\Delta}p_x = \frac{\left[1-\Delta\frac{\left(a+e^{b+cx}+e^{b+c(x+\Delta)}\right)}{4}\right]}{\left[1+\Delta\frac{\left(a+e^{b+cx}+e^{b+c(x+\Delta)}\right)}{4}\right]} = \frac{\left[4-\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} \quad (92) $$

Recall $ {}_{\Delta}p_x = e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)} $ (93)

$$ {}_{\Delta}q_x = \left\{ \begin{array}{l} 1-\frac{\left[4-\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} = \\ \frac{4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)-\left[4-\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} \end{array} \right\} \quad (94) $$

Death probability $ {}_{\Delta}q_x = \frac{4\Delta a+\Delta e^{b+cx}+2\Delta e^{b+c(x+\Delta)}+\Delta e^{b+cx}}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} $ (95)

Equating (92) and (93)

$$ e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)} = \frac{\left[4-\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} \quad (96) $$

$$ Error_3 = e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)} - \frac{\left[4-\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]}{\left[4+\Delta\left(2a+e^{b+cx}+e^{b+c(x+\Delta)}\right)\right]} \quad (97) $$

Q.E.D

2. Materials and Methods

In actuarial life mathematics and mortality, interest rate is assumed constant and consequently, the following life insurance basic annuity integral theorem shall be stated and proved to allow us make our derivations.

Theorem

$$\bar{a}_x = \int_0^{\Omega-x} (1+i)^{-t} \frac{l_{x+t}}{l_x} dt = \int_0^{\Omega-x} \left(\lim_{K\to\infty} \left(1+\frac{i^{(K)}}{K}\right)^K \right)^{-t} \frac{l_{x+t}}{l_x} dt \tag{98}$$

where ${}_t p_x = \frac{l_{x+t}}{l_x}$ and $l_x = \int_0^{\infty} l_{x+t}\mu_{x+t} dt$ is the expected number of lives surviving to age x

Proof

Let σ be the force of interest**,** i be the valuation interest rate and K be a positive integer representing the number of times interest rates is compounded

Common textbooks in theory of interest define $i+1 = \left(1+\frac{i^{(K)}}{K}\right)^K$ (99)

$$\bar{a}_x = \int_0^{\Omega-x} \begin{pmatrix} 1+K\times\left(\frac{i^{(K)}}{K}\right)^1 + \frac{K(K-1)}{2!}\times\left(\frac{i^{(K)}}{K}\right)^2 \\ +\frac{K(K-1)(K-2)}{3!}\times\left(\frac{i^{(K)}}{K}\right)^3 + \ldots \end{pmatrix}^{-t} \frac{l_{x+t}}{l_x} dt \tag{100}$$

where Ω is the maximum age in a mortality table.

$$\bar{a}_x = \int_0^{\Omega-x} \begin{pmatrix} 1+K\times\left(\frac{i^{(K)}}{K}\right)^1 + \frac{K(K-1)}{K^2}\times\frac{1}{2!}\left(i^{(K)}\right)^2 \\ +\frac{K(K-1)(K-2)}{K^3}\times\frac{1}{3!}\left(i^{(K)}\right)^3 + \ldots \end{pmatrix}^{-t} \frac{l_{x+t}}{l_x} dt \tag{101}$$

$$\bar{a}_x = \int_0^{\Omega-x} \left(\left\{ \lim_{K\to\infty} \left(1 + \frac{i^{(K)}}{K} \right)^K \right\} \right)^{-t} \frac{l_{x+t}}{l_x} dt \tag{102}$$

$$\bar{a}_x = \int_0^{\Omega-x} \left(\lim_{K\to\infty} \left\{ \begin{array}{l} 1 + i^{(K)} + \dfrac{K(K-1)}{K^2} \times \dfrac{1}{2!} \left(i^{(K)} \right)^2 \\ + \dfrac{K(K-1)(K-2)}{K^3} \times \dfrac{1}{3!} \left(i^{(K)} \right)^3 + \ldots \end{array} \right\} \right)^{-t} \frac{l_{x+t}}{l_x} dt \tag{103}$$

$$\bar{a}_x = \int_0^{\Omega-x} \left(\begin{array}{l} 1 + i^{(K)} + \dfrac{K^2 \left(1 - \dfrac{1}{K} \right)}{K^2} \times \dfrac{1}{2!} \left(i^{(K)} \right)^2 \\ + \dfrac{K^3 \left(1 - \dfrac{1}{K} \right) \left(1 - \dfrac{2}{K} \right)}{K^3} \times \dfrac{1}{3!} \left(i^{(K)} \right)^3 + \ldots \end{array} \right)^{-t} \frac{l_{x+t}}{l_x} dt \tag{104}$$

$$\bar{a}_x = \int_0^{\Omega-x} \left(1 + i^{(\infty)} + \frac{1}{2!} \left(i^{(\infty)} \right)^2 + \frac{1}{3!} \left(i^{(\infty)} \right)^3 + \frac{1}{4!} \left(i^{(\infty)} \right)^4 + \ldots \right)^{-t} \frac{l_{x+t}}{l_x} dt \tag{105}$$

$$\bar{a}_x = \int_0^{\Omega-x} \left(1 + i^{(\infty)} + \frac{1}{2!} \left(i^{(\infty)} \right)^2 + \frac{1}{3!} \left(i^{(\infty)} \right)^3 + \frac{1}{4!} \left(i^{(\infty)} \right)^4 \ldots \right)^{-t} \left({}_t p_x \right) dt \tag{106}$$

$$\bar{a}_x = \int_0^{\Omega-x} \left(1 + \frac{\sigma}{1!} + \frac{\sigma^2}{2!} + \frac{\sigma^3}{3!} + \frac{\sigma^4}{4!} + \ldots \right)^{-t} \left({}_t p_x \right) dt \tag{107}$$

But e^{σ} is equal to the inside bracket in equation (106) and (105)

This implies that $\sigma = i^{(\infty)}$ and consequently,

$$\bar{a}_x = \int_0^{\Omega-x} \left(e^{\sigma} \right)^{-t} \left({}_t p_x \right) dt \tag{108}$$

$$\bar{a}_x = \int_0^{\Omega-x} e^{-\sigma t} \left({}_t p_x \right) dt \tag{109}$$

QED

Using equation (1) and the definitions below it, we obtain

$$\bar{a}_x = \int_0^{\Omega-x} \zeta^t \delta^{H^x\left(H^t-1\right)} e^{-\sigma t} dt \tag{110}$$

$$\left({}_t p_x\right) e^{-\sigma t} = e^{-\sigma t}\zeta^t \delta^{H^x\left(H^t-1\right)} = e^{-\sigma t}\zeta^t \delta^{\left(H^{x+t}-H^x\right)} = \frac{e^{-\sigma t}\zeta^t \delta^{\left(H^{x+t}\right)}}{\delta^{H^x}} \tag{111}$$

$${}_t p_x e^{-\sigma t} = e^{-\sigma t}\zeta^t \delta^{H^x\left(H^t-1\right)} = \frac{e^{-\sigma t}\zeta^t \delta^{\left(H^{x+t}\right)}}{\delta^{H^x}} = \frac{\exp\left(H^t H^x \log_e \delta + t\log_e\left(e^{-\sigma}\zeta\right)\right)}{\delta^{H^x}} \tag{112}$$

Observe that $H^t = e^{\log_e H^t} = e^{t\log_e H}$ (113)

$$\begin{aligned} {}_t p_x e^{-\sigma t} &= e^{-\sigma t}\zeta^t \delta^{H^x\left(H^t-1\right)} = \frac{e^{-\sigma t}\zeta^t g^{\left(H^{x+t}\right)}}{\delta^{H^x}} \\ &= \frac{\exp\left(\left(e^{t\log_e H}\right)\left(H^x \times \log_e g\right) + t\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}} \end{aligned} \tag{114}$$

$$\bar{a}_x = \int_0^{\Omega-x} \frac{\exp\left(\left(e^{t\log_e H}\right)\left(H^x \times \log_e \delta\right) + t\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}} dt \tag{115}$$

Let $\eta = t\log_e H \Rightarrow \dfrac{\eta}{\log_e H} = t$ (116)

When $t = 0$, $\eta = 0$

When $t = \Omega - x$, $\eta = \left(\Omega - x\right)\log_e H = \log_e H^{(\Omega-x)}$ (117)

$$\frac{d\eta}{dt} = \log_e H \Rightarrow d\eta = \log_e H dt \Rightarrow \frac{d\eta}{\log_e H} = dt \tag{118}$$

$$\bar{a}_x = \int_0^{\log_e H^{(\Omega-x)}} \frac{\exp\left(e^{\eta}\left(H^x \times \log_e \delta\right) + \frac{\eta}{\log_e H} \times \log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}} \frac{d\eta}{\log_e H} \tag{119}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left(\left(e^{\eta} \times H^x \times \log_e \delta\right) + \frac{\eta}{\log_e H} \times \log_e e^{-\sigma}\zeta\right) d\eta \tag{120}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left(\left(H^x \times \log_e \delta\right) e^{\eta} + \left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \tag{121}$$

Following Gradshteyn and Ryzhik (2007 pp. 356, formula, $ETI147(37)$)

$$\int_0^{\infty} \exp\left(-\alpha e^{Y} - \bar{\delta}Y\right) dY = \alpha^{\delta}\Gamma\left(-\bar{\delta}, \alpha\right) \tag{122}$$

where $\Gamma(.)$ is the gamma function

$$\alpha = a + ib,\ i = \sqrt{-1} \text{ and } a > 0 \text{ and } a \le \operatorname{Re}\left|\alpha\right| \tag{123}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \tag{124}$$

$$\int_0^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta = \left\{\begin{array}{l} \displaystyle\int_0^{\log_e H^{(\Omega-x)}} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \\ \displaystyle + \int_{\log_e H^{(\Omega-x)}}^{\infty} \exp\left(-\left(-H^x \times \log_e \delta\right) e^{\eta} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right) d\eta \end{array}\right\} \tag{125}$$

Consequently, $$\bar{a}_x = \frac{\left\{ \begin{array}{l} \int_0^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\eta \right) d\eta - \\ \int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\eta \right) d\eta \end{array} \right\}}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \quad (126)$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \left\{ \begin{array}{l} \int_0^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\eta \right) d\eta - \\ \int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\eta \right) d\eta \end{array} \right\} \quad (127)$$

$$\begin{array}{l} L_1 = \int_0^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\eta \right) d\eta = \\ \left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e \zeta}{\log_e H}\right)} \Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right), \left(-H^x \times \log_e \delta\right) \right) \end{array} \quad (128)$$

$$L_2 = \int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^\eta - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\eta \right) d\eta \quad (129)$$

Let $\xi = \eta - \log_e H^{\Omega-x}$ (130)

Let $\xi + \log_e H^{\Omega-x} = \eta$ (131)

$$d\xi = d\eta \quad (132)$$

When $\eta = \infty$, $\xi = \infty$

When $\eta = \log_e H^{\Omega-x}$, $\xi = 0$

Therefore,

$$L_2 = \int_0^\infty \exp\left(-\left(-H^x \times \log_e \delta\right)e^{\xi + \log_e H^{\Omega-x}} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H} \right)\left(\xi + \log_e H^{\Omega-x}\right) \right) d\xi \quad (133)$$

$$L_2 = \int_0^\infty \exp\left\{\begin{array}{l} -\left(-H^x \times \log_e \delta\right) e^{\log_e H^{\Omega-x}} e^{\xi} \\ -\left(\left(-\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi + \left(\left(-\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\right)\log_e H^{\Omega-x}\right) \end{array}\right\} d\xi \tag{134}$$

$$L_2 = \int_0^\infty \exp\left\{\begin{array}{l} -\left(-H^x \times \log_e \delta\right) H^{\Omega-x} e^{\xi} \\ -\left(-\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi - \left(-\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)(\Omega - x)(\log_e H) \end{array}\right\} d\xi \tag{135}$$

$$L_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right) H^{\Omega} e^{\xi} + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi - \left(-\log_e e^{-\sigma}\zeta\right)(\Omega - x)\right\} d\xi \tag{136}$$

$$L_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right) H^{\Omega} e^{\xi} + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi + \log_e \zeta^{\Omega-x}\right\} d\xi \tag{137}$$

$$L_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right) H^{\Omega} e^{\xi} + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi + \log_e e^{-\sigma(\Omega-x)}\right\} d\xi \tag{138}$$

$$L_2 = \int_0^\infty \exp\left\{-\left(-\log_e \delta\right) H^{\Omega} e^{\xi} + -\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi - \sigma(\Omega - x)\right\} d\xi \tag{139}$$

$$L_2 = e^{-\sigma(\Omega-x)} \int_0^\infty \exp\left\{-\left(-\log_e \delta\right) H^{\Omega} \times e^{\xi} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi\right\} d\xi \tag{140}$$

$$\begin{array}{l} L_2 = e^{-\sigma(\Omega-x)} \displaystyle\int_0^\infty \exp\left\{-\left(-\log_e \delta\right) H^{\Omega} \times e^{\xi} - \left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\xi\right\} d\xi = \\ e^{-\sigma(\Omega-x)} \left[\left(-\log_e \delta\right) H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)} \Gamma\left(-\left(-\dfrac{\log_e e^{-\sigma}\zeta}{\log_e H}\right), \left(-\log_e \delta\right) H^{\Omega}\right) \end{array} \tag{141}$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \left(J_1 - J_2\right) \tag{142}$$

$$\bar{a}_x =$$

$$\frac{1}{(\log_e H)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-H^x \times \log_e \delta\right)\right)- \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-\log_e \delta\right)H^{\Omega}\right)\end{array}\right\}$$

(143)

$$\bar{a}_x = \frac{1}{(\log_e H)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-H^x \times \log_e \delta\right)\right)- \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-\log_e \delta\right)H^{\Omega}\right)\end{array}\right\}$$

(144)

where $H^x = \frac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2}$ (145)

$$\log_e \zeta = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)}}{s}\right] \Rightarrow \zeta = e^{\left(\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s-1\right)}}{s}\right)} \quad (146)$$

$$\log_e \delta = \frac{\alpha\left(H^s - 1\right)^{-2}}{H^x} \Rightarrow H^x \log_e \delta = \alpha\left(H^s - 1\right)^{-2} \Rightarrow \log_e \delta^{H^x} = \alpha\left(H^s - 1\right)^{-2} \quad (147)$$

$$\Rightarrow \delta^{H^x} = e^{\alpha\left(H^s - 1\right)^{-2}}$$

$$H^x = \frac{\alpha}{\log_e \delta}\left(H^s - 1\right)^{-2} \quad (148)$$

$$\bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x \times \log_e \delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-H^x \times \log_e \delta\right)\right)- \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e \delta\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-\log_e \delta\right)H^{\Omega}\right)\end{array}\right\}$$

(149)

2.1. Application to Pension Fund Using Commutation Functions

Let ϕ be the benefits paid out continuously by a life office to a scheme holder. Suppose an annuity holder retiring at age 70 buys annuity policy at age 50. We assume that the scheme holder survives till 55 years.

By definition $\bar{N}_x = \int_0^{\infty} D_{x+\xi} d\xi$ and $\bar{a}_x = \frac{1}{D_x}\int_0^{\infty} D_{x+\xi} d\xi$

At time ξ, the cash outflow from life office is defined as

$$B(\xi) = \left\{\frac{\bar{N}_{55} - \bar{N}_{70}}{D_{\xi}(50)}\right\} \times \phi = \frac{\phi}{D_{\xi}(50)} \times \int_{55}^{70} D_{\zeta} d\zeta \tag{150}$$

And the cash inflow to the life office is

$$A(\xi) = \left\{\frac{\bar{N}_{50} - \bar{N}_{65}}{D_{\xi}(50)}\right\} \times \pi = \frac{\pi}{D_{\xi}(50)} \times \int_{50}^{65} D_{\zeta} d\zeta \tag{151}$$

Both $B(\xi)$ and $A(\xi)$ are functions depending on age x at time ξ. The total number of policy holders aged 50 is l_{50}

$$B(\xi) = l_{50} \times \left\{\frac{\bar{N}_{55} - \bar{N}_{70}}{D_{\xi}(50)}\right\} \times \phi = \frac{\phi \times l_{50}}{D_{\xi}(50)} \times \int_{55}^{70} D_{\zeta} d\zeta \tag{152}$$

$$A(\xi) = l_{50} \times \left\{\frac{\bar{N}_{50} - \bar{N}_{65}}{D_{\xi}(50)}\right\} \times \pi = \frac{\pi \times l_{50}}{D_{\xi}(50)} \times \int_{50}^{65} D_{\zeta} d\zeta \tag{153}$$

$$\frac{\bar{N}_{55}-\bar{N}_{70}}{D_{\xi}(50)}=\left\{\begin{array}{l}\frac{\bar{N}_{55}-\bar{N}_{56}}{D_{\xi}(50)}+\frac{\bar{N}_{56}-\bar{N}_{57}}{D_{\xi}(50)}+\frac{\bar{N}_{57}-\bar{N}_{58}}{D_{\xi}(50)}\times \\ \frac{\bar{N}_{58}-\bar{N}_{59}}{D_{\xi}(50)}+\frac{\bar{N}_{59}-\bar{N}_{60}}{D_{\xi}(50)}+\ldots+\frac{\bar{N}_{69}-\bar{N}_{70}}{D_{\xi}(50)}\end{array}\right\} \tag{154}$$

$$\begin{aligned}&\frac{\bar{N}_{55}-\bar{N}_{70}}{D_{\xi}(50)}= \\ &\frac{1}{D_{\xi}(50)}\left\{\int_{55}^{56} D_{\zeta} d\zeta+\int_{56}^{57} D_{\zeta} d\zeta+\int_{57}^{58} D_{\zeta} d\zeta+\int_{58}^{59} D_{\zeta} d\zeta+\int_{59}^{60} D_{\zeta} d\zeta+\ldots+\int_{69}^{70} D_{\zeta} d\zeta\right\}\end{aligned} \tag{155}$$

$$\frac{\bar{N}(55)-\bar{N}(70)}{D_{\xi}(50)}=\frac{1}{D_{\xi}(50)}\left\{\sum_{i=56}^{70} \int_{i}^{i+1} D_{\zeta} d\zeta\right\} \tag{156}$$

$$\bar{a}_{x:\overline{n}|}=\int_{0}^{n} v^{t}\left({}_{t}P_{x}\right) dt=\frac{\bar{N}_{x}-\bar{N}_{x+n}}{D_{x}} \Rightarrow D_{x} \times \bar{a}_{x:\overline{n}|}=\bar{N}_{x}-\bar{N}_{x+n} \tag{157}$$

$$\bar{a}_{55:\overline{15}|}=\int_{0}^{15} v^{t}\left({}_{t}P_{x}\right) dt=\frac{\bar{N}_{55}-\bar{N}_{55+17}}{D_{55}} \Rightarrow D_{55} \times \bar{a}_{55:\overline{15}|}=\bar{N}_{55}-\bar{N}_{55+15} \tag{158}$$

$$\begin{aligned}&\frac{1}{D_{\xi}(50)}\left\{\sum_{i=56}^{70} \int_{i}^{i+1} D_{\zeta} d\zeta\right\}=\frac{D_{55}}{D_{\xi}(50)} \times \bar{a}_{55:\overline{15}|}=\frac{l_{55} v^{55}}{l_{\xi+50} v^{\xi+50}} \times \bar{a}_{55:\overline{15}|} \\ &=\frac{l_{55}}{l_{\xi+50}} \times v^{5-\xi} \times \bar{a}_{55:\overline{15}|}\end{aligned} \tag{159}$$

$$\frac{1}{v^{50+\xi} l_{\xi+50}}\left\{\sum_{i=56}^{70} \int_{i}^{i+1} D_{\zeta} d\zeta\right\}=\frac{l_{55}}{l_{\xi+50}} \times v^{5-\xi} \times \bar{a}_{55:\overline{15}|} \tag{160}$$

$$\bar{a}_{55:\overline{15}|}=\frac{v^{-55}}{l_{55}}\left\{\sum_{i=56}^{70} \int_{i}^{i+1} D_{\zeta} d\zeta\right\}=\frac{1}{v^{55} l_{55}}\left\{\sum_{i=56}^{70} \int_{i}^{i+1} D_{\zeta} d\zeta\right\}=\frac{1}{D_{55}}\left\{\sum_{i=56}^{70} \int_{i}^{i+1} D_{\zeta} d\zeta\right\} \tag{161}$$

For ease of computation, equation (149) was broken down into segments as shown in columns $1-10$ and in column 11 the continuous life annuity is computed

2.2. Data Analysis and Implementation.

Many economies have cautiously developed various financially sustainable pension programs for their citizens. However, the ideal pension plans for retirees vary from economy to economy. The Makeham's framework will enable us to estimate the life annuity as applicable in pensions administration. Having derived the life annuity formulae based on $\mu_x = \rho + GH^x$, current survival data from the US social security tables were used to estimate the parameters using advanced algebraic method as $\rho = 0.0002462199082$; $G = 0.00001878394898$ and $H = 1.102973888$.

Table 1 below shows the systematic computation of life annuities based on equation (149)

Table1: Table of Life Annuity $\bar{a}_x$

1 x	2	3	4	5	6
30	0.094629	0.000034	0.000697	2.838859	17.096240
31	0.094629	0.000034	0.000697	2.933487	18.793060
32	0.094629	0.000034	0.000697	3.028116	20.658280
33	0.094629	0.000034	0.000697	3.122745	22.708620
34	0.094629	0.000034	0.000697	3.217373	24.962470
35	0.094629	0.000034	0.000697	3.312002	27.440000
36	0.094629	0.000034	0.000697	3.406631	30.163440
37	0.094629	0.000034	0.000697	3.501259	33.157180
38	0.094629	0.000034	0.000697	3.595888	36.448040
39	0.094629	0.000034	0.000697	3.690516	40.065530
40	0.094629	0.000034	0.000697	3.785145	44.042060
41	0.094629	0.000034	0.000697	3.879774	48.413260
42	0.094629	0.000034	0.000697	3.974402	53.218300

43	0.094629	0.000034	0.000697	4.069031	58.500250
44	0.094629	0.000034	0.000697	4.163660	64.306430
45	0.094629	0.000034	0.000697	4.258288	70.688870
46	0.094629	0.000034	0.000697	4.352917	77.704780
47	0.094629	0.000034	0.000697	4.447545	85.417030
48	0.094629	0.000034	0.000697	4.542174	93.894710
49	0.094629	0.000034	0.000697	4.636803	103.213800
50	0.094629	0.000034	0.000697	4.731431	113.457800
51	0.094629	0.000034	0.000697	4.826060	124.718600
52	0.094629	0.000034	0.000697	4.920689	137.097000
53	0.094629	0.000034	0.000697	5.015317	150.703900
54	0.094629	0.000034	0.000697	5.109946	165.661400
55	0.094629	0.000034	0.000697	5.204574	182.103400
56	0.094629	0.000034	0.000697	5.299203	200.177200
57	0.094629	0.000034	0.000697	5.393832	220.044900
58	0.094629	0.000034	0.000697	5.488460	241.884500
59	0.094629	0.000034	0.000697	5.583089	265.891700
60	0.094629	0.000034	0.000697	5.677718	292.281600
61	0.094629	0.000034	0.000697	5.772346	321.290700
62	0.094629	0.000034	0.000697	5.866975	353.178900
63	0.094629	0.000034	0.000697	5.961603	388.232100
64	0.094629	0.000034	0.000697	6.056232	426.764400

65	0.094629	0.000034	0.000697	6.150861	469.121000
66	0.094629	0.000034	0.000697	6.245489	515.681500
67	0.094629	0.000034	0.000697	6.340118	566.863200
68	0.094629	0.000034	0.000697	6.434747	623.124700
69	0.094629	0.000034	0.000697	6.529375	684.970100
70	0.094629	0.000034	0.000697	6.624004	752.953800
71	0.094629	0.000034	0.000697	6.718632	827.684900
72	0.094629	0.000034	0.000697	6.813261	909.833100
73	0.094629	0.000034	0.000697	6.907890	1000.134000
74	0.094629	0.000034	0.000697	7.002518	1099.398000
75	0.094629	0.000034	0.000697	7.097147	1208.514000
76	0.094629	0.000034	0.000697	7.191776	1328.460000
77	0.094629	0.000034	0.000697	7.286404	1460.310000
78	0.094629	0.000034	0.000697	7.381033	1605.247000
79	0.094629	0.000034	0.000697	7.475662	1764.569000
80	0.094629	0.000034	0.000697	7.570290	1939.703000
81	0.094629	0.000034	0.000697	7.664919	2132.220000
82	0.094629	0.000034	0.000697	7.759547	2343.844000
83	0.094629	0.000034	0.000697	7.854176	2576.471000
84	0.094629	0.000034	0.000697	7.948805	2832.187000
85	0.094629	0.000034	0.000697	8.043433	3113.284000
86	0.094629	0.000034	0.000697	8.138062	3422.279000

87	0.094629	0.000034	0.000697	8.232691	3761.942000
88	0.094629	0.000034	0.000697	8.327319	4135.317000
89	0.094629	0.000034	0.000697	8.421948	4545.749000
90	0.094629	0.000034	0.000697	8.516576	4996.917000
91	0.094629	0.000034	0.000697	8.611205	5492.864000
92	0.094629	0.000034	0.000697	8.705834	6038.033000
93	0.094629	0.000034	0.000697	8.800462	6637.312000
94	0.094629	0.000034	0.000697	8.895091	7296.068000
95	0.094629	0.000034	0.000697	8.989720	8020.207000

Table 1: Continued

7	8	9	10	11 $\bar{a}_x = 8 \times 9 \times 10$
0.125884	11.990000	0.151363	4.758511	8.632553
0.138378	12.140000	0.164991	4.233562	8.476990
0.152112	12.300000	0.179847	3.758239	8.316225
0.167209	12.490000	0.196040	3.328409	8.150358
0.183805	12.700000	0.213691	2.940274	7.979516
0.202047	12.930000	0.232931	2.590337	7.803857
0.222101	13.200000	0.253904	2.275385	7.623570
0.244144	13.490000	0.276765	1.992457	7.438875
0.268376	13.820000	0.301684	1.738825	7.250024
0.295012	14.190000	0.328847	1.511978	7.057299

0.324292	14.620000	0.358455	1.309597	6.861016
0.356479	15.090000	0.390730	1.129543	6.661518
0.391859	15.640000	0.425910	0.969839	6.459180
0.430751	16.260000	0.464258	0.828660	6.254403
0.473504	16.970000	0.506059	0.704313	6.047615
0.520499	17.780000	0.551623	0.595235	5.839264
0.572159	18.730000	0.601290	0.499975	5.629822
0.628946	19.820000	0.655428	0.417187	5.419774
0.691369	21.100000	0.714441	0.345624	5.209622
0.759988	22.600000	0.778768	0.284128	4.999873
0.835418	24.370000	0.848886	0.231625	4.791041
0.918333	26.470000	0.925318	0.187118	4.583642
1.009478	29.000000	1.008632	0.149683	4.378184
1.109670	32.060000	1.099446	0.118468	4.175169
1.219805	35.790000	1.198438	0.092683	3.975085
1.340871	40.390000	1.306342	0.071605	3.778400
1.473953	46.140000	1.423962	0.054570	3.585561
1.620244	53.410000	1.552172	0.040974	3.396988
1.781054	62.730000	1.691926	0.030273	3.213071
1.957825	74.860000	1.844263	0.021977	3.034166
2.152140	90.920000	2.010316	0.015651	2.860592
2.365741	112.570000	2.191320	0.010916	2.692631

2.600542	142.360000	2.388622	0.007442	2.530522
2.858647	184.280000	2.603687	0.004949	2.374464
3.142369	244.730000	2.838117	0.003203	2.224615
3.454251	334.300000	3.093654	0.002012	2.081088
3.797088	471.010000	3.372200	0.001224	1.943957
4.173951	686.600000	3.675824	0.000718	1.813257
4.588218	1039.000000	4.006787	0.000406	1.688982
5.043601	1638.270000	4.367548	0.000220	1.571093
5.544181	2702.620000	4.760792	0.000113	1.459516
6.094444	4685.550000	5.189442	0.000056	1.354145
6.699321	8579.370000	5.656687	0.000026	1.254850
7.364232	16681.030000	6.166001	0.000011	1.161474
8.095136	34645.780000	6.721173	0.000005	1.073841
8.898583	77371.820000	7.326332	0.000002	0.991756
9.781772	187131.550000	7.985977	0.000001	0.915010
10.752620	494060.530000	8.705015	0.000000	0.843384
11.819820	1.44E+06	9.488794	0.000000	0.776648
12.992950	4.64E+06	10.343140	0.000000	0.714570
14.282500	1.69E+07	11.274410	0.000000	0.656914
15.700050	6.96E+07	12.289530	0.000000	0.603441
17.258290	3.30E+08	13.396060	0.000000	0.553918
18.971180	1.83E+09	14.602200	0.000000	0.508112

20.854080	1.20E+10	15.916950	0.000000	0.465795
22.923860	9.54E+10	17.350080	0.000000	0.426748
25.199070	9.29E+11	18.912230	0.000000	0.390756
27.700090	1.13E+13	20.615050	0.000000	0.357614
30.449340	1.77E+14	22.471180	0.000000	0.327125
33.471450	3.63E+15	24.494430	0.000000	0.299102
36.793510	1.01E+17	26.699850	0.000000	0.273366
40.445280	3.88E+18	29.103840	0.000000	0.249748
44.459500	2.15E+20	31.724280	0.000000	0.228090
48.872130	1.77E+22	34.580650	0.000000	0.208242
53.722720	2.27E+24	37.694210	0.000000	0.190063
59.054720	4.69E+26	41.088110	0.000000	0.173423

3. Results and Discussion

In table1, it is observed from the computational results that the trend of annuities progressively reduces downwards from ages $30-90$. This trend justifies the mathematical behavior that $\bar{a}_x$ is a decreasing function. To demonstrate how long the annuitants are covered by the decreasing life insurance scheme, the table 1 presents the terms of the deceasing whole life annuity scheme across the ages. The older annuitants will pay a higher percentage of premium for the whole life insurance scheme and consequently can only enjoy a shorter term of insurance coverage since at senescence, annuitants earn higher payments and have a higher mortality rate. The smaller the interest rate becomes, the bigger the proportion the of the term insurance and smaller costs for the life annuity hence proportions of the mortality component are usually higher than old annuitants.

Define $\gamma(Z,x)=\int_0^x t^{Z-1}e^{-t}dt$ as the incomplete lower Gamma integral

$\Gamma(Z,x)=\int_{x}^{\infty}t^{Z-1}e^{-t}dt$ as the incomplete upper Gamma integral

$\Gamma(Z)=\int_{0}^{\infty}t^{Z-1}e^{-t}dt$ as the Gamma integral and the exponential integral function is defined as follows

$$E_m(x)=\int_{1}^{\infty}e^{-xt}t^{-m}dt=\left\{1+\frac{m}{(x+m)^2}+\frac{m(m-2x)}{(x+m)^4}+\frac{m\left(6x^2-8mx+m^2\right)}{(x+m)^6}+R(m,x)\right\}$$

(162)

where $-36m^{-4}\leq 100R(m,x)\leq\left(100+\frac{100}{x+m-1}\right)m^{-4}$ (163)

We can now compute the lower Gamma integral as

$$\gamma(Z,x)=\int_{0}^{x}e^{-t}t^{Z-1}dt \tag{164}$$

$$\gamma(Z,x)=\int_{0}^{x}\left(1-t+\frac{t^2}{2}-\frac{t^3}{6!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}+...\right)t^{Z-1}dt \tag{165}$$

$$\gamma(Z,x)=\int_{0}^{x}t^{Z-1}-t^{Z}+\frac{t^{Z+1}}{2}-\frac{t^{Z+2}}{3!}+\frac{t^{Z+3}}{4!}+\frac{t^{Z+4}}{5!}+\frac{t^{Z+5}}{6!}+...dt \tag{166}$$

$$\gamma(Z,x)=\left[\frac{t^Z}{z}-\frac{t^{Z+1}}{z+1}+\frac{t^{Z+2}}{2(z+2)}-\frac{t^{Z+3}}{3!\times(z+3)}+\frac{t^{Z+4}}{4!(z+4)}+\frac{t^{Z+5}}{5!(z+5)}+\frac{t^{Z+6}}{6!(z+6)}+...\right]_0^t$$

(167)

$$\gamma(Z,x)=\frac{t^Z}{z}-\frac{t^{Z+1}}{z+1}+\frac{t^{Z+2}}{2(z+2)}-\frac{t^{Z+3}}{3!\times(z+3)}+\frac{t^{Z+4}}{4!(z+4)}+\frac{t^{Z+5}}{5!(z+5)}+\frac{t^{Z+6}}{6!(z+6)}+...$$

(168)

$$\gamma(Z,x)=\sum_{k=0}^{\infty}(-1)^k\frac{t^{Z+k}}{k!(z+k)} \tag{169}$$

$$\Gamma(Z)=\int_{0}^{\infty}t^{Z-1}e^{-t}dt \tag{170}$$

$$\Gamma\left(Z,x\right)=\int_{0}^{\infty}t^{Z-1}e^{-t}dt-\int_{0}^{x}t^{Z-1}e^{-t}dt \tag{171}$$

$$\Gamma\left(Z,x\right)=\Gamma\left(Z\right)-\gamma\left(Z,x\right) \tag{172}$$

$$\Gamma\left(Z,x\right)=\Gamma\left(Z\right)-\sum_{k=0}^{\infty}\left(-1\right)^{k}\frac{t^{Z+k}}{k!\left(z+k\right)} \tag{173}$$

$$\Gamma\left(Z,x\right)=\Gamma\left(Z\right)\left(\Gamma\left(Z\right)-\sum_{k=0}^{\infty}\left(-1\right)^{k}\frac{t^{Z+k}}{k!\left(z+k\right)}\right) \tag{174}$$

Substituting equations (145)-(148) into (149), we obtain

$$\bar{a}_x=\frac{\left\{\begin{array}{l}\left[\left(-H^{x}\times\frac{\alpha\left(H^{s}-1\right)^{-2}}{H^{x}}\right)\right]^{\left(-\frac{\log_e e^{-\sigma}e^{\left(\frac{\log_e l_{x+s}-\log_e l_x-\frac{\alpha}{\left(H^{s}-1\right)}}{s}\right)}}{\log_e H}\right)}\Gamma\left(\begin{array}{l}\left(\frac{\log_e e^{-\sigma}e^{\left(\frac{\log_e l_{x+s}-\log_e l_x-\frac{\alpha}{\left(H^{s}-1\right)}}{s}\right)}}{\log_e H}\right),\\ \left(-H^{x}\times\frac{\alpha\left(H^{s}-1\right)^{-2}}{H^{x}}\right)\end{array}\right)-\\ e^{-\sigma\left(\Omega-x\right)}\left[\left(-\frac{\alpha\left(H^{s}-1\right)^{-2}}{\frac{\alpha}{\log_e\delta}\left(H^{s}-1\right)^{-2}}\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}e^{\left(\frac{\log_e l_{x+s}-\log_e l_x-\frac{\alpha}{\left(H^{s}-1\right)}}{s}\right)}}{\log_e H}\right)}\Gamma\left(\begin{array}{l}\left(\frac{\log_e e^{-\sigma}e^{\left(\frac{\log_e l_{x+s}-\log_e l_x-\frac{\alpha}{\left(H^{s}-1\right)}}{s}\right)}}{\log_e H}\right)\\ ,\left(-\frac{\alpha\left(H^{s}-1\right)^{-2}}{\frac{\alpha}{\log_e\delta}\left(H^{s}-1\right)^{-2}}\right)H^{\Omega}\end{array}\right)\end{array}\right\}}{\left(\log_e H\right)\left(e^{\alpha\left(H^{s}-1\right)^{-2}}\right)} \tag{175}$$

$$\bar{a}_x=\frac{\left\{\begin{array}{l}\left[\left(-\alpha\left(H^s-1\right)^{-2}\right)\right]^{\left(-\frac{\log_e e^{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)-\sigma}}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)-\sigma}}{\log_e H}\right),\left(-\alpha\left(H^s-1\right)^{-2}\right)\right)-\\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e\delta\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}-\sigma}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}-\sigma}{\log_e H}\right),\left(-\log_e\delta\right)H^{\Omega}\right)\end{array}\right\}}{\left(\log_e H\right)\left(e^{\alpha\left(H^s-1\right)^{-2}}\right)} \tag{176}$$

$$\bar{a}_x=\frac{\left\{\begin{array}{l}\left[\left(-\alpha\left(H^s-1\right)^{-2}\right)\right]^{\left(-\frac{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)-\sigma}{\log_e H}\right)}\Gamma\left(\left(\frac{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)-\sigma}{\log_e H}\right),\left(-\alpha\left(H^s-1\right)^{-2}\right)\right)-\\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e\delta\right)H^{\Omega}\right]^{\left(-\frac{e^{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}-\sigma}{\log_e H}\right)}\Gamma\left(\left(\frac{\left(\frac{\log_e\left({}_s p_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)-\sigma}{\log_e H}\right),\left(-\log_e\delta\right)H^{\Omega}\right)\end{array}\right\}}{\left(\log_e H\right)\left(e^{\alpha\left(H^s-1\right)^{-2}}\right)} \tag{177}$$

$$\lim_{\sigma\to 0}\bar{a}_x = \bar{e}_x \tag{178}$$

Consequently, the life expectancy becomes

$$\bar{e}_x = \frac{\left\{\begin{array}{l}\left[\left(-\alpha\left(H^s-1\right)^{-2}\right)\right]^{\left(-\frac{\left(\frac{\log_e\left({}_sP_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}{\log_e H}\right)}\Gamma\left(\left(\frac{\left(\frac{\log_e\left({}_sP_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}{\log_e H}\right),\left(-\alpha\left(H^s-1\right)^{-2}\right)\right)- \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e\delta\right)H^{\Omega}\right]^{\left(-\frac{e^{\left(\frac{\log_e\left({}_sP_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}-\sigma}{\log_e H}\right)}\Gamma\left(\left(\frac{\left(\frac{\log_e\left({}_sP_x\right)-\frac{\alpha}{\left(H^s-1\right)}}{s}\right)}{\log_e H}\right),\left(-\log_e\delta\right)H^{\Omega}\right)\end{array}\right\}}{\left(\log_e H\right)\left(e^{\alpha\left(H^s-1\right)^{-2}}\right)} \tag{179}$$

$$\lim_{Z\to 0}\Gamma\left(Z,x\right)=\lim_{Z\to 0}\int_x^{\infty}t^{Z-1}e^{-t}dt = E_1\left(x\right) \tag{180}$$

where $E_1\left(x\right)=-\gamma-\log_e x-\sum_{r=1}^{\infty}\frac{\left(-1\right)^r x^r}{r\times\Gamma\left(r+1\right)}$ (181)

γ is the Euler's constant and $\Gamma\left(r+1\right)=r!$.

Consequently, we can express the $GM\left(1,2\right)$ as $\mu_x=\rho+GH^x=\rho+Ge^{\phi x}$

Again, following Dragan (2022), when $0 < Ge^{\phi x} \leq \phi$ and $0 < 10\rho \leq \phi$, then the incomplete Gamma function can be approximated by

$$\Gamma(Z,x) = \left(\frac{Z}{Z+Z^2} e^{\left[(1-\gamma)Z + \frac{Z^2}{2} \times \left(\sum_{u=1}^{\infty} \frac{1}{u^2} - 1 \right)\right]} \right) - \left(\sum_{r=0}^{\infty} \frac{(-1)^r x^{Z+r}}{r!(Z+r)} \right) \tag{182}$$

$$\Gamma(Z,x) = \left(\frac{Z}{Z+Z^2} e^{\left[(1-\gamma)Z + (0.3225) \times Z^2\right]} \right) - \left(\sum_{r=0}^{\infty} \frac{(-1)^r x^{Z+r}}{r!(Z+r)} \right) \tag{183}$$

where $\left(\sum_{u=1}^{\infty} \frac{1}{u^2} - 1 \right) = 2 \times (0.3225)$ (184)

and $\varsigma(2) = \sum_{u=1}^{\infty} \frac{1}{u^2}$, $\varsigma(k) = \sum_{u=1}^{\infty} u^{-k}$ is the Riemann-Zeta function

4. Conclusion

We have constructed a mathematical framework to explain the characteristics of the continuous life annuity function under the assumption of the actuarial equivalence principle. The pension fund can then adapt it to derive the monthly payments for given level of contributions under the option of lump-sum amount deposited to the pension funds. After indicating the assumptions of mortality and approved valuation of interest rate in the annuity model, we show clear arguments of estimations through Gradshteyn and Ryzhik (2007) integral. We then examined the behavior of life annuity scheme in terms of the proportion of the decreasing insurance components, the annuity rate in terms of the present value random variable. In order to provide a mathematical framework to measure the effects of mortality risks on an annuity plan, contributions for guaranteed benefits as well as the actuarial reserve in subsequent 66+years can be computed under the following modification technique for pension funds such as $\prod = m \times \Phi \times \left(\ddot{a}_x^{(m)} \right) \left[(100+r) \times \prod \times \left(A^{1\ (m)}_{x:\overline{\xi}|} \right) - m \times \Phi \times \left(I^m A \right)^{1\ (m)}_{x:\overline{\xi}|} \right]$ in life insurance set up particularly when the management of pension funds examines longevity risks of varying degrees about insured group. In such a modification, the annuity plan may offer to guarantee each annuitant to earn a total payout amounting to $(100+r)\%$ of the premium contribution $\prod$ for as long as the annuitant survives, where $0 < r < 1$ and Φ is the *mthly* sum payable to the insured annuitants. For future directions, stochastic methods can be applied to address the risk of uncertainties.

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