Research Article

The Predicted Yield Curve of Interest Rates Involving Bjork and Christensen Four Factor Model

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Abstract

Modelling the term structure of interest rate has become very important in recent days because the predictability of the variables induces informed decision to various market stakeholders, regulatory authorities and investors who might need such information. A deep knowledge of this model's structure is essential in appraising the interest rate risk of financial institutions since investment decisions resulting from liquidity gaps have effect on interest rate risk. The study contributes to the asset-liability administration of Nigerian financial institutions thereby providing a solution to the cost effective approximation of defined term structures problem which are applied in establishing fund transfer pricing mechanisms. Therefore, investment houses require estimations of interest rates so as to price derivative instruments and for pension funds, future interest rate are of keen interest to estimate the value of their assets and liabilities. The objectives of the study are (i) To predict the in-sample of the Nigerian Eurobond under the Bjork and Christensen four-factor model (ii) To derive the console rate and short term rate and show the mathematical relationship between the rates (iii) To investigate whether time to maturity has an effect on term structure of interest rate. (iv) To derive the yield function from the forward rate function. The approach used to estimate the model parameters is the ordinary least square method. The resulting computed parameters were used to estimate the in-sample yield after it was being substituted in the model. Test of goodness of fit was conducted and the result reveals that the model fits in well with the observed data yield and this was demonstrated by the model's -square adjusted by comparing yield curve between the observed and predicted yields.

***Keywords**: Bjork and Christensen four-factor model, Predictability, term structure of interest rate, yield curve model*

1. Introduction

Term structure of interest rates defines the mathematical relationship between yield to maturity of risk free zero coupon bond and their maturities. Consequently, it is described by means of a yield curve indicating varying rate at different maturities. In (De Pooter, 2007) it is confirmed that nine out of thirteen Central Banks that report their curve estimation methods to the Bank of International Settlements use the Nelson-Siegel class or their variants of which Bjork and Christensen is one. The European Central Bank used the extended Nelson-Siegel model to estimate the term structure. Following (Ibanez, 2015; Wahlstrom, *at al.*, 2021, Atsushi, 2022), the yield curve can assume different trajectories associated with upward sloping, downward sloping, flat or humped shaped. From the financial mathematics literature, it can be observed that all term structure models can be categorized as either spline based technique or parametric based technique. The current structure is a function based model which falls under the Nelson-Siegel's class of parametric models since they are categorized as single valued functions that are defined over the entire maturity domain. Essentially it is developed to estimate one of the three equivalent forms of the term structure tools: Spot rate, discount rate or forward rate function. Nigeria's bond market representing a major component of the capital market and an important means of monetary transmission framework has suffered continuous liquidity problems. A bond produces a negative yield when the price that investors pay for it is greater than the interest and principal they will get back over its life. Investors may be willing to accept this loss in exchange for the relative safety that a borrower such as a fiscally strong government or a major corporation provides and consequently, this scenario occurs when the economy is weak. The inadequacy of debt instruments have been responsible for the poor performance of the capital market operations. In capital market, there are many kinds of investments of which zero coupon bond is one and the interest rate earned on this bond is the yield. The yield represents the market expectation and at same time relies on the interest rate trajectories based on the market price at a definite time. The objectives of the study are (i) To predict the in-sample of the Nigerian Eurobond under the Bjork and Christensen four-factor model (ii) To derive the console rate and short term rate and show the mathematical relationship between the rates (iii) To investigate whether time to maturity has an effect on term structure of interest rate. (iv) To derive the yield function from the forward rate function. The study contributes to the asset and liability administration of financial institutions in Nigeria by providing a solution to the estimation of defined term structures which are applied in setting up fund transfer pricing mechanisms. In (Gasha, *at al.*, 2010; Hokuto, 2019), an appropriate estimation of the term structure of interest rates represents a core function in actuarial and financial risk management such that monetary policymakers find it useful when advising government. Following (Vasicek, 1977; Ranik *at al.*, 2021), as the projection of the economic variables arouses informed decision to appropriate policy makers who would need such information, the yield curve is employed to appraise the effect of fiscal policies across the whole economy provided the current interest rate and the implied forward rate curves are known. This may consist of ensuring adequacy of derivative pricing and hedging, projecting the yields over long-term instruments and enhancing the monetary policies and debt policies, consequently, adequate techniques of computing the yield curves should be established that could be applied to support fiscal decisions. Following (Due and Rui, 1996; Evans and Marshall, 2001; Diebold and Canlin, 2006; Ishii, 2018; Jan, *at al.*, 2023), the term structure of interest rates at definite maturity horizons define core inputs into the valuation of various investible instruments such that quantitative representation of interest rate risk will be markedly important to investors. Since investment managers constantly employ interest rate spreads between short-term and risky rates as a measuring instrument to measure the relative liquidity risk and credit worthiness, a deep knowledge of the drivers of interest rates trajectories and the determinants of bond returns would seem important for fixed-income investors. Experts investigate yield curve models that concentrate on the dynamics of the term structure to the extent that the desire for such models is aroused by the increasing need to price appropriately over long term horizons interest rate derivatives such that in attaining this objective, it is necessary to model the yield curves and the volatility of interest rates when they occur over time. In (Ilmanem and Iwanowski, 1997; Wu, 2003; Rajna, *at al.*, 2010 and Christensen, *at al.*, 2011; Nymand-Anderson, 2018), by reason of the rapid spread of term structure models which have been suggested previously, the phenomena seems to be theoretically and slowly investigated because it falls within the center of valuation problems experienced in actuarial finance. Usually, various financial assets are valued by discounting the expected future cash flows to the present provided a suitable discount rate function which encapsulates within an underlying phenomena on term structure is known.

1.1. Mathematical preliminaries of term structure of interest rate

A zero-coupon bond commencing at the time τ and maturing at time S is a security bounded by a legal promise from the party issuing it to pay one unit of money to the bondholder at maturity time. The price $p(\tau,S)$ for $\tau < S$ reaches the maximum at time S and consequently as $\tau \to S$, $p(S,S)=1$

The yield to maturity $Z(\tau,S)$ at the time τ of a zero coupon bond defines the continuously compounded rate of return through which the price of a zero-coupon bond $p(\tau,S)$ progressively rises from time τ to time S so as to accrue 1 unit of money at time S and consequently,

$$p(\tau,S)e^{Z(\tau,S)(S-\tau)}=1 \tag{1}$$

$$p(\tau,S)=e^{-Z(\tau,S)(S-\tau)} \tag{2}$$

$$\log_e p(\tau,S)=\log_e e^{-Z(\tau,S)(S-\tau)} \Rightarrow \log_e p(\tau,S)=-Z(\tau,S)(S-\tau) \tag{3}$$

$$Z(\tau,S)=-\frac{\log_e p(\tau,S)}{(S-\tau)}=\frac{1}{(S-\tau)}\log_e\frac{1}{\left[p(\tau,S)\right]} \tag{4}$$

Suppose that $p(\tau)$ represents the time τ market value of a fixed coupon bond with coupon payments dates arranged as $S_1<S_2<S_3<...<S_m$ having corresponding coupons $\alpha_1,\alpha_2,\alpha_3,...,\alpha_m$ together with nominal investment M, then by reason of (2), the market price at time τ of the zero coupon bond is expressed by

$$p(\tau)=M\times p(\tau,S_m)+\sum_{j=1}^{m}\alpha_j p(\tau,S_j) \tag{5}$$

$$p(\tau)=M\times e^{-Z(\tau,S_m)(S_m-\tau)}+\sum_{j=1}^{m}\alpha_j e^{-Z(\tau,S_j)(S_j-\tau)}\ \text{ for } \tau\le S_1 \tag{6}$$

where $p(\tau,S_j)$ represent the discount factors mapped into the coupon payment α_j while $Z(\tau,S_j)$ defines the continuously compounded yield described by (4)

The instantaneous spot rate $R(\tau)$ at time τ represents the yield function on the currently maturing bond. Using equation (4), we have

$$R(\tau)=\lim_{S\to\tau}Z(\tau,S)=Z(\tau,\tau)=1 \tag{7}$$

The instantaneous spot rate function is the rate of return received by investors within the subsequent short interval of time. From equation (4), the yield curve defines the function $S\to Z(\tau,S)$ such that at time τ defines the functional relationship between the bond's yields and their respective time to maturity. Considering an investor who at time τ holds a bond having maturity at time $S_1 > \tau$ whose price is $p(\tau, S_1)$ and wishes to roll it forward to the subsequent equivalent period of time $S_2 > S_1$ to a fixed rate $f(\tau, S_1, S_2)$ that is agreed upon presently. This implies investing at time τ in a bond maturing at time S_2 and trading for the price $p(\tau, S_2)$. This means that the forward rates are interest rates or the rate of return that are locked in now for an investment in a future time such that they are set consistently with the current term structure of discount factors. These rates are obtained from the equations

$$p(\tau, S_1) = p(\tau, S_2) e^{(S_2 - S_1)\times f(\tau, S_1, S_2)} \tag{8}$$

which holds for any pair of maturities $S_j < S_k$

From (8), $f(\tau, S_1, S_2)$ becomes

$$f(\tau, S_1, S_2) = \frac{1}{S_2 - S_1} \log_e \left[\frac{p(\tau, S_1)}{p(\tau, S_2)} \right] \tag{9}$$

Using equations (2) we have

$$f(\tau, S_1, S_2) = \frac{1}{S_2 - S_1} \log_e \left[\frac{e^{-Z(\tau, S_1)(S_1 - \tau)}}{e^{-Z(\tau, S_2)(S_2 - \tau)}} \right] \tag{10}$$

$$f(\tau, S_1, S_2) = \frac{1}{S_2 - S_1} \log_e e^{Z(\tau, S_2)(S_2 - \tau) - Z(\tau, S_1)(S_1 - \tau)} \tag{11}$$

$$f(\tau, S_1, S_2) = \frac{Z(\tau, S_2)(S_2 - \tau) - Z(\tau, S_1)(S_1 - \tau)}{S_2 - S_1} \tag{12}$$

This is the rate of return for an investment on a forward contract entered at time τ but starting at time S_1 and provides payment at time S_2

In defining the instantaneous forward rate $f(\tau,S)$, we need to set $S=S_1$ and let

$S_2 \to S$ and hence $f(\tau,S)=\lim\limits_{T_2\to T^+} f(\tau,S,S_2)=-\dfrac{\partial \log_e p(\tau,S)}{\partial S}$ (13)

Consequently, the forward rate function is defined as $S \to f(\tau,S)$ and defines the graph of forward rate for all maturities. From equation (4)

$$Z(\tau,S)(S-\tau)=-\log_e p(\tau,S) \tag{14}$$

$$\frac{\partial \log_e p(\tau,S)}{\partial S}=-\frac{\partial}{\partial S}\log_e\left[Z(\tau,S)(S-\tau)\right] \tag{15}$$

Consequently,

$$f(\tau,S)=\lim_{T_2\to T^+} f(\tau,S,S_2)=-\frac{\partial}{\partial S}\left(\log_e p(\tau,S)\right)=\frac{\partial}{\partial S}\log_e\left[Z(\tau,S)(S-\tau)\right] \tag{16}$$

Therefore given $f(\tau,S)$ for $0\le\tau\le S$

$$\int_\tau^S f(\tau,\xi)\,d\xi=-\left[\log_e p(\tau,S)-\log_e p(\tau,\tau)\right] \tag{17}$$

where $p(\tau,\tau)=1$ (18)

$$\int_\tau^S f(\tau,\xi)\,d\xi=-\left[\log_e p(\tau,S)-\log_e 1\right] \tag{19}$$

$$\int_\tau^S f(\tau,\xi)\,d\xi=-\log_e p(\tau,S) \tag{20}$$

$$p(\tau,S)=e^{-\int_{\tau}^{S} f(\tau,\xi)d\xi} \tag{21}$$

$$y(\tau,S)=\frac{1}{S-\tau}\int_{\tau}^{S} f(\tau,\xi)d\xi \tag{22}$$

Thus, by the mean value theorem for integrals, the continuously compounded spot rate defines the average of the forward rates occurring between the times τ and S

2. Material and Methods

The historical end-of-day average bid task price and yield quotes for the Nigerian Eurobond from January to December of the year 2020 was obtained from Nigerian debt Management Board which was quoted by Bloomberg. The maturities obtained and used are one year, two years, three years, five years, seven years, ten years, eleven years, twelve years, eighteen years, twenty seven years and twenty nine years. The (Nelson and Siegel, 1987) model which was formulated by (Diebol and Li, 2006) was modified with the goal of fitting the forward rate curve at a given date as an approximating function. The Nelson-Siegel model initially presumes that the forward rate function follows a matrix function of the form

$$[\beta_0,\beta_1,\beta_2]\begin{bmatrix}1\\ e^{-\lambda_t\tau}\\ \tau\lambda e^{-\lambda_t\tau}\end{bmatrix}=[\beta_0,\beta_1,\beta_2]\begin{bmatrix}f_0\\ f_1\\ f_2\end{bmatrix} \tag{23}$$

The model consists of a constant f_0 and an exponentially decay function f_1 derived from the second order differential equation of the form

$$\beta_2\frac{d^2 f}{dt^2}+\beta_1\frac{df}{dt}+\beta_0 f=0 \tag{24}$$

denoted here by f_2. The forward rate function can be functionally expressed in the form

$$\beta_0 f_0+\beta_1 f_1+\beta_2 f_2 \tag{25}$$

$$f_\tau(\tau)=\beta_{0t}+\beta_{1t}\exp(-\lambda_t\tau)+\beta_{2t}\tau\lambda_t\exp(-\lambda_t\tau) \tag{26}$$

where $\beta_{0t};\beta_{1t};\beta_{2t};\lambda_t$ are the varying parameters.
The yield curve (yield as a function of maturity) can also be derived as below

By definition $Z(\tau,S)=\frac{1}{S-\tau}\int_\tau^S f(\tau,\xi)\,d\xi$ (27)

$$Z(\tau,S)=\frac{1}{S}\int_0^S f(\tau,\xi)\,d\xi \tag{28}$$

Equation (28) could be written as $Z_\tau(\tau)=\frac{1}{\tau}\int_0^\tau f_\tau(u)\,du$ (29)

$$Z_\tau(\tau)=\frac{1}{\tau}\int_0^\tau\left\{\beta_{0t}+\beta_{1t}\exp(-\lambda_t u)+\beta_{2t}\lambda_t u\exp(-\lambda_t u)\right\}du \tag{30}$$

Solving the above function we have the spot rate as

$$Z_\tau(\tau)=\beta_{0t}+\beta_{1t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right)+\beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}-e^{-\lambda_t\tau}\right) \tag{31}$$

where $\beta_{0t};\beta_{1t};\beta_{2t};\lambda_t$ are parameters. When we take the limiting values of the function $y_\tau(\tau)$ as τ tending to infinity, the resulting value is β_{0t} and when we tend the function to zero, the resulting values is $\beta_{0t}+\beta_{1t}$ (32)

Hence, the consol rate $\lim_{\tau\to\infty}Z_\tau(\tau)=\beta_{0t}$ (33)

Thus, we can posit that β_{0t} is the long rate.

We calculate the limit of the function tending to zero as follows:

$$\lim_{\tau\to 0}Z_\tau(\tau)=\lim_{\tau\to 0}\left[\beta_{0t}+\beta_{1t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right)+\beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}-e^{-\lambda_t\tau}\right)\right] \tag{34}$$

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left[\lim_{\tau\to 0} \beta_{0t} + \lim_{\tau\to 0} \left(\beta_{1t} + \beta_{2t}\right) \left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau} \right) - \lim_{\tau\to 0} \beta_{2t} e^{-\lambda_t \tau} \right] \tag{35}$$

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left[\lim_{\tau\to 0} \beta_{0t} + \left(\beta_{1t} + \beta_{2t}\right) \lim_{\tau\to 0} \left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau} \right) - \lim_{\tau\to 0} \beta_{2t} e^{-\lambda_t \tau} \right] \tag{36}$$

Applying the L'Hopital's hypothesis to the second term, we have

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left[\lim_{\tau\to 0} \beta_{0t} + \left(\beta_{1t} + \beta_{2t}\right) \lim_{\tau\to 0} \left(\frac{\lambda_t e^{-\lambda_t \tau}}{\lambda_t} \right) - \lim_{\tau\to 0} \beta_{2t} e^{-\lambda_t \tau} \right] \tag{37}$$

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left[\lim_{\tau\to 0} \beta_{0t} + \left(\beta_{1t} + \beta_{2t}\right) \lim_{\tau\to 0} \left(e^{-\lambda_t \tau} \right) - \lim_{\tau\to 0} \beta_{2t} e^{-\lambda_t \tau} \right] \tag{38}$$

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left[\beta_{0t} + \left(\beta_{1t} + \beta_{2t}\right) - \beta_{2t} \right] \tag{39}$$

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left[\beta_{0t} + \beta_{1t} \right] \tag{40}$$

This implies that the short rate is $\left[\beta_{0t} + \beta_{1t}\right]$

Following (Nelson and Siegel, 1987) the shape flexibility could be alternatively explained by interpreting the parameters of (31) as defining the strengths of the short-term, medium-term and long-term function of the forward rate curve. β_{0t} contributes to the long-term function, the contribution of the short-term function is β_{1t} while β_{2t} shows the contribution of the
medium-term function. Apparently, the medium-term function within the model commences at zero and exponentially decays to zero. The short-term function seems to be the fastest decaying functions within the model and decays monotonically to zero. To increase the flexibility of the Nelson Siegel model, (Bjork and Christensen, 1999) modified the model by introducing an additional factor that made it a four-factor model.

2.1. Model specification (Bjork and Christensen)

In the study we seek to analyze and model the term structure of interest rate of the Nigerian Eurobond. In order to achieve this objective, this study adopts the Bjork and Christensen four-factor model. The yield function through Bjork and Christensen's model is described as follows: Observe that the forward function is given as

$$f_\tau(\tau)=\beta_{1t}+\beta_{2t}\exp(-\frac{\tau}{\lambda_\tau})+\beta_{3,t}(\frac{\tau}{\lambda_\tau})\exp(-\frac{\tau}{\lambda_\tau})+\beta_{4,t}\exp(-\frac{2\tau}{\lambda_\tau}) \qquad (41)$$

Then the yield function is

$$Z_\tau(\tau)=\beta_{1t}+\beta_{2t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}\right)+\beta_{3,t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{4,t}\left(\frac{1-e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2\tau}{\lambda_t}}\right)$$

where $\beta_{1\tau}$ $\beta_{2\tau}$ $\beta_{3\tau}$ $\beta_{4\tau}$ and λ_τ are parameters.

Using equation (26), the yield curve function could be derived by integrating the forward rate function.

$$Z_\tau(\tau)=\frac{1}{\tau}\int_0^\tau \beta_{1t}+\beta_{2t}\exp(-\frac{u}{\lambda_\tau})+\beta_{3,t}(\frac{u}{\lambda_\tau})\exp(-\frac{u}{\lambda_\tau})+\beta_{4,t}\exp(-\frac{2u}{\lambda_\tau})du \quad (42)$$

$$Z_\tau(\tau)=\frac{1}{\tau}\int_0^\tau \beta_{1t}+\beta_{2t}e^{-\frac{u}{\lambda_\tau}}+\beta_{3,t}\frac{u}{\lambda_\tau}e^{-\frac{u}{\lambda_\tau}}+\beta_{4,t}e^{-\frac{2u}{\lambda_\tau}}du \qquad (43)$$

$$Z_\tau(\tau)=\frac{1}{\tau}\left\{\begin{array}{l}\left[\beta_{1t}u\right]_0^\tau-\left[\lambda_t\beta_{2t}e^{-\frac{u}{\lambda_\tau}}\right]_0^\tau+\left(\left[-\beta_{3,t}\frac{1}{\lambda_\tau}u\lambda_t e^{-\frac{u}{\lambda_\tau}}\right]_0^\tau-\int_0^\tau-\beta_{3,t}\frac{1}{\lambda_\tau}\lambda_t e^{-\frac{u}{\lambda_\tau}}du\right)\\ -\left[\frac{\lambda_t}{2}\beta_{4,t}e^{-\frac{2u}{\lambda_\tau}}\right]_0^\tau\end{array}\right\} (44)$$

$$Z_{\tau}(\tau)=\frac{1}{\tau}\left\{\begin{array}{l}\left[\beta_{1t}u\right]_0^{\tau}-\left[\lambda_t\beta_{2t}e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}+\left[-\beta_{3,t}\frac{1}{\lambda_\tau}u\lambda_t e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}+\int_0^{\tau}\beta_{3,t}\frac{1}{\lambda_\tau}\lambda_t e^{-\frac{u}{\lambda_\tau}}du\\-\left[\frac{\lambda_t}{2}\beta_{4,t}e^{-\frac{2u}{\lambda_\tau}}\right]_0^{\tau}\end{array}\right\} \quad (45)$$

$$Z_{\tau}(\tau)=\frac{1}{\tau}\left\{\begin{array}{l}\left[\beta_{1t}u\right]_0^{\tau}-\left[\lambda_t\beta_{2t}e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}+\left[-\beta_{3,t}\frac{1}{\lambda_\tau}u\lambda_t e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}-\left[\beta_{3,t}\frac{1}{\lambda_\tau}\lambda_t\lambda_t e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}\\-\left[\frac{\lambda_t}{2}\beta_{4,t}e^{-\frac{2u}{\lambda_\tau}}\right]_0^{\tau}\end{array}\right\} \quad (46)$$

$$Z_{\tau}(\tau)=\frac{1}{\tau}\left\{\left[\beta_{1t}u\right]_0^{\tau}-\left[\lambda_t\beta_{2t}e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}+\left[-\beta_{3,t}ue^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}-\left[\beta_{3,t}\lambda_t e^{-\frac{u}{\lambda_\tau}}\right]_0^{\tau}-\left[\frac{\lambda_t}{2}\beta_{4,t}e^{-\frac{2u}{\lambda_\tau}}\right]_0^{\tau}\right\} \quad (47)$$

$$Z_{\tau}(\tau)=\frac{1}{\tau}\left\{\begin{array}{l}\beta_{1t}\tau-\left[\lambda_t\beta_{2t}e^{-\frac{\tau}{\lambda_\tau}}-\lambda_t\beta_{2t}\right]-\left[\beta_{3,t}\tau e^{-\frac{\tau}{\lambda_\tau}}\right]-\left[\beta_{3,t}\lambda_t e^{-\frac{\tau}{\lambda_\tau}}-\beta_{3,t}\lambda_t\right]\\-\left[\frac{\lambda_t}{2}\beta_{4,t}e^{-\frac{2\tau}{\lambda_\tau}}-\frac{\lambda_t}{2}\beta_{4,t}\right]\end{array}\right\} \quad (48)$$

$$Z_{\tau}(\tau)=\beta_{1t}+\beta_{2t}\left(\frac{\lambda_t}{\tau}-\frac{\lambda_t}{\tau}e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{3,t}\left(\frac{\lambda_t}{\tau}-\frac{\lambda_t}{\tau}e^{-\frac{\tau}{\lambda_\tau}}-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{4,t}\left(\frac{\lambda_t}{2\tau}-\frac{\lambda_t}{2\tau}e^{-\frac{2\tau}{\lambda_\tau}}\right) \quad (49)$$

$$Z_{\tau}(\tau)=\beta_{1t}+\frac{\lambda_t}{\tau}\beta_{2t}\left(1-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{3,t}\left(\frac{\lambda_t}{\tau}\left(1-e^{-\frac{\tau}{\lambda_\tau}}\right)-e^{-\frac{\tau}{\lambda_\tau}}\right)+\frac{\lambda_t}{2\tau}\beta_{4,t}\left(1-e^{-\frac{2\tau}{\lambda_\tau}}\right) \quad (50)$$

Thus, the yield function becomes

$$Z_\tau(\tau)=\beta_{1t}+\beta_{2t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}\right)+\beta_{3,t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{4,t}\left(\frac{1-e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2\tau}{\lambda_t}}\right) \quad (51)$$

Q.E.D

Theorem1

If the yield function is

$$Z_\tau(\tau)=\beta_{1t}+\beta_{2t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}\right)+\beta_{3,t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{4,t}\left(\frac{1-e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2\tau}{\lambda_t}}\right)$$

Then the consol is $\lim\limits_{\tau\to\infty} Z_\tau(\tau)=\beta_{1t}$

Proof

$$\lim_{\tau\to\infty} Z_\tau(\tau)=\lim_{\tau\to\infty}\left\{\beta_{1t}+\beta_{2t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}\right)+\beta_{3,t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{4,t}\left(\frac{1-e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2\tau}{\lambda_t}}\right)\right\} \quad (52)$$

Applying L'Hopital rule

$$\lim_{\tau\to\infty} Z_\tau(\tau)=\lim_{\tau\to\infty}\left\{\beta_{1t}+\beta_{2t}\left(\frac{\frac{1}{\lambda_t}e^{-\frac{\tau}{\lambda_\tau}}}{\frac{1}{\lambda_t}}\right)+\beta_{3,t}\left(\frac{\frac{1}{\lambda_t}e^{-\frac{\tau}{\lambda_\tau}}}{\frac{1}{\lambda_t}}-e^{-\frac{\tau}{\lambda_\tau}}\right)+\beta_{4,t}\left(\frac{\frac{2}{\lambda_t}e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2}{\lambda_t}}\right)\right\} \quad (53)$$

$$\lim_{\tau\to\infty} Z_\tau(\tau)=\beta_{1t} \quad (54)$$

Q.E.D

Theorem 2

If the yield function is

$$Z_\tau(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}\right) + \beta_{3,t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}} - e^{-\frac{\tau}{\lambda_\tau}}\right) + \beta_{4,t}\left(\frac{1-e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2\tau}{\lambda_t}}\right)$$

The short term rate is $\lim_{\tau\to 0} Z_\tau(\tau) = \beta_{1t} + \beta_{2t} + \beta_{4,t} = \textbf{consol} + \beta_{2t} + \beta_{4,t}$

Proof

$$\lim_{\tau\to 0} Z_\tau(\tau) = \lim_{\tau\to 0}\left\{\beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}}\right) + \beta_{3,t}\left(\frac{1-e^{-\frac{\tau}{\lambda_\tau}}}{\frac{\tau}{\lambda_t}} - e^{-\frac{\tau}{\lambda_\tau}}\right) + \beta_{4,t}\left(\frac{1-e^{-\frac{2\tau}{\lambda_\tau}}}{\frac{2\tau}{\lambda_t}}\right)\right\} \quad (55)$$

Applying L'Hopital's rule

$$\lim_{\tau\to 0} Z_\tau(\tau) = \left\{\beta_{1t} + \beta_{2t}\left(\frac{\frac{1}{\lambda_t}}{\frac{1}{\lambda_t}}\right) + \beta_{3,t}\left(\frac{\frac{1}{\lambda_t}}{\frac{1}{\lambda_t}} - 1\right) + \beta_{4,t}\left(\frac{\frac{2}{\lambda_t}}{\frac{2}{\lambda_t}}\right)\right\} \quad (56)$$

$$\lim_{\tau\to 0} Z_\tau(\tau) = \beta_{1t} + \beta_{2t} + \beta_{4,t} \quad (57)$$

The short term rate is $\lim_{\tau\to 0} Z_\tau(\tau) = \textbf{consol} + \beta_{2t} + \beta_{4,t}$

Q.E.D

2.2. Method of data analysis

We also employ the ordinary least square method to analyze the Bjork and Christensen model given the observed data. Essentially, in order to obtain Lamda following (Diebold and Canlin, 2006), the third term in equation (51) can be expressed as $\bar{\lambda} = \arg\max_{\lambda}\left[\frac{1-e^{\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right]$ where $\lambda = 0.0609$ and such that the loading of the curvature factor achieves its maximum for a maturity of 2.5 years which is usually observed as the medium term.

The data presented in this study involves the daily closing of the Nigerian Eurobond which comprises January to December 2020 to analyze and fit the Nigerian Eurobond yield curve using the Bjork and Christensen (1999) four-factor model. The data for the whole year 2020 were analyzed and the findings were also depicted in graphs and tabular form to include other statistic not captured on the curve and also to enhance easy access to the statistic numerical values.

Aggregate Descriptive Analysis

The daily yields of the Nigerian Eurobond were extracted and analyzed based on descriptive analysis in order to have the necessary statistics for further analysis and also for an informed decision. The overall descriptive data is presented in table 1 which will be used to analyze the aggregate model as presented below:

Table 1: Aggregate Descriptive Statistics

Maturity	N	Minimum	Maximum	Mean	Std. Deviation	Variance
1 years	250	-1.0690	16.8150	5.052452	4.1617513	17.320
2 years	250	2.2750	14.2740	5.623268	2.9418536	8.655
3 years	250	2.8480	15.1730	5.916892	2.9572819	8.746
5 years	250	4.1230	15.8540	7.132140	2.6467443	7.005
7 years	250	5.1320	14.1940	7.533984	2.1501411	4.623
10 years	250	5.9680	13.7050	8.004524	1.8366449	3.373
11 years	250	6.5020	15.7820	8.635112	2.0687794	4.280
12 years	250	6.5320	15.2730	8.541268	1.8034079	3.252
18 years	250	6.9940	13.4140	8.649296	1.4474288	2.095
27 years	250	7.1230	13.0210	8.620676	1.2674708	1.606
29 years	250	.0000	14.7490	9.206624	1.6376774	2.682

Source: author’s computation via SPSS 23

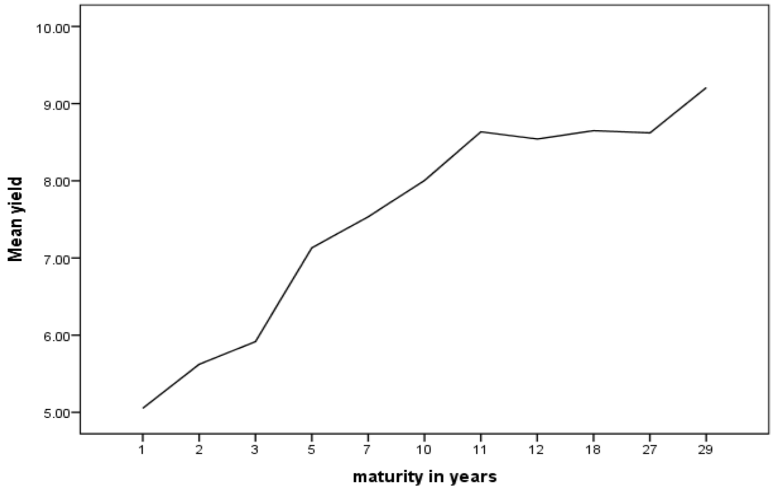

Figure 1: Aggregate Nigerian Eurobond Yield curve 2020

From the above table1, the overall descriptive statistics presented eleven tenors. This includes one year, two years, three years, five years, seven years, ten years eleven years, twelve years, eighteen years, twenty seven years and twenty nine years with respective average yield of

$$5.052452,\ 5.623268,\ 5.916892,\ 7.132140,\ 7.533984,$$
$$8.004524,\ 8.635112,\ 8.541268,\ 8.649296, 8.620676\ \mathit{and}\ 9.206624$$

respectively. From table 1 and figure one, it is observed that the aggregate yield curve is upward sloping with a little decline on twenty seven years maturity and by this, we can infer that the longer the maturity of a security, the higher its expected yield. Also observed is the negative yield recorded on the one year maturity as shown on the minimum yield. There is also a higher volatility with one year having the highest while the twenty seven years maturity have the lowest volatility.

2.3. How Do We Predict the In-Sample Yield of τ ?

The (Bjork and Christensen, 1999) four-factor model can be used to estimate in-sample of τ (τ as time to maturity) given an observed yield, the model parameters were estimated using the ordinary least square method. The result is presented in table 1.

Table 2: The Beta Coefficients

`		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	β_1	9.432	.204		46.149	.000
	β_2	-10.450	14.629	-1.595	-.714	.498
	β_3	-2.131	7.601	-.137	-.280	.787
	β_4	6.742	16.883	.744	.399	.702

Source: author's computation via SPSS 23

The parameters as estimated in the above table 2 can be substituted for the estimation of f in-sample maturities that were not captured on the observed data. Since the in-sample yield does not exceed a maturity of more than twenty nine years, we write the conditional model for in-sample estimation of yield as follows:

$$y_\tau(\tau) = 9.432 - 10.450\left(\frac{1-e^{-\frac{\tau}{0.0609}}}{\frac{\tau}{0.0609}}\right) - 2.131\left(\frac{1-e^{-\frac{\tau}{0.0609}}}{\frac{\tau}{0.0609}} - e^{-\frac{\tau}{0.0609}}\right) + 6.742\left(\frac{1-e^{-\frac{2\tau}{0.0609}}}{\frac{2\tau}{0.0609}}\right) \quad (58)$$

for $0 \le \tau \le 348$ (59)

The figure 2 below shows the model yield curve and the observed yield curve

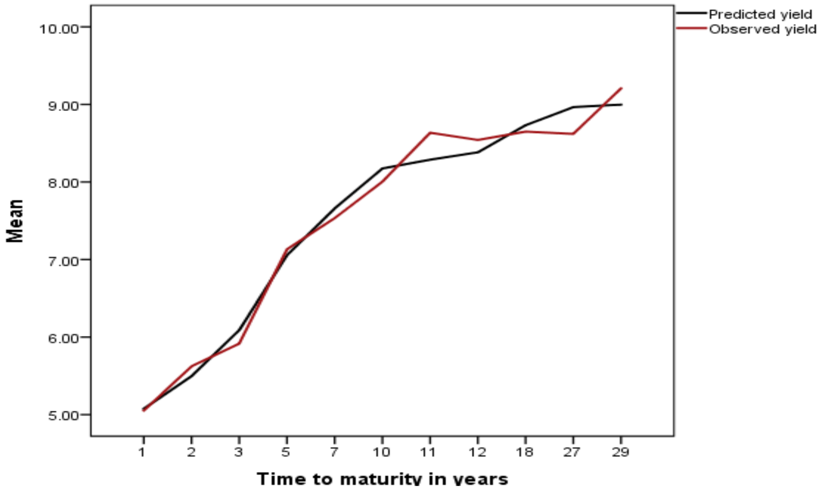

Figure 2: Observed and Predicted yield curve

2.4 How Does the Model Fit into the Observed Data?

The measure goodness of fit is determined when ordinary lease square method is applied on data by R^2 adjusted. The model measure of fit analysis is depicted in table 3.

Table 4: Model summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.990[a]	.980	.971	.242769635

Source: author's computation via SPSS 23

From the above table, the R square (R^2) is 0.980 and the R square adjusted is 0.971 with the standard error of the estimate of 0.242769635. Using the adjusted R Square as our measure of goodness of fit, we can say that the observed data can be explained by the model with 97.1 % degree of accuracy. Therefore, we can conclude that the model fit well on the observed data and can be used for further studies or purposes

2.5. Does Time to Maturity Have an Effect on Term Structure of Interest Rate?

The relationship that existed between time to maturity of a set of a bond and its corresponding yield is referred to as the term structure of interest rate. To determine whether time to maturity has an effect on the term structure of interest rate, we used correlation between time to maturity and its corresponding as presented in table 4

Table 3: Correlations

		Yield	Maturity
Yield	Pearson Correlation	1	.826**
	Sig. (2-tailed)		.002
	N	11	11
Maturity	Pearson Correlation	.826**	1
	Sig. (2-tailed)	.002	
	N	11	11

Source: author’s computation via SPSS 23

The above table 4 of the Pearson correlation reveals that the correlation is significant at 0.01 level. Therefore, we conclude that time to maturity has an effect on term structure of interest rate.

3. Results and Discussion

In table1, it was revealed that as time to maturity increases, the yield also increases. This shows that the yield of Nigerian Eurobond is directly proportional to its time to maturity since the slope is positive. However, the aggregate descriptive statistics confirms a positive slope. The analysis of the Bjork and Christensen’s four-factor model reveals that the in-sample maturities that were not captured on the observed data can be estimated with constraint of 348 months. The analysis of goodness of fit of the (Bjork and Christensen, 1999) four-factor model shows that the model fits in well into the observed data. This is indicated by R square adjusted with the value as 0.971 which means that $97.1\ \%$ of the observed data can be explained by the model. The predicted yields for β_1, β_2, β_3 and β_4 are shown in figures tables 5, tables 6 ,tables 7, tables 8 while the predicted yield curves are depicted in figures 3, figures 4, figures 5 and figures 6

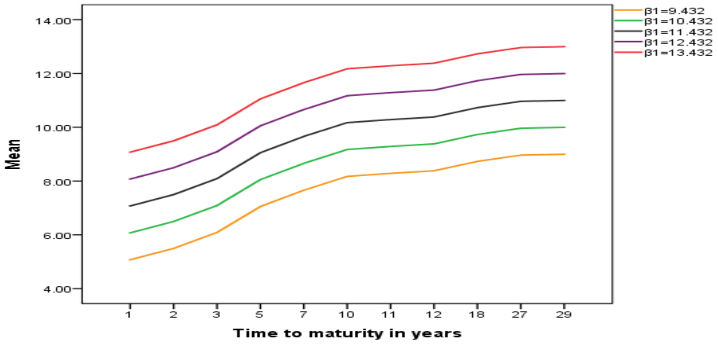

Figure 3: Predicted Yield using values of β_1

Table 5: Predicted yields using different values of β_1

Time to maturity	β_1=9.432	β_1=10.432	β_1=11.432	β_1=12.432	β_1=13.432
1 year	5.075579	6.075579	7.075579	8.075579	9.075579
2 years	5.496644	6.496644	7.496644	8.496644	9.496644
3 years	6.090609	7.090609	8.090609	9.090609	10.09061
5 years	7.055196	8.055196	9.055196	10.0552	11.0552
7 years	7.65919	8.65919	9.65919	10.65919	11.65919
10 years	8.174346	9.174346	10.17435	11.17435	12.17435
11 years	8.287524	9.287524	10.28752	11.28752	12.28752
12 years	8.382358	9.382358	10.38236	11.38236	12.38236
18 years	8.731875	9.731875	10.73188	11.73188	12.73188
27 years	8.965246	9.965246	10.96525	11.96525	12.96525
29 years	8.997436	9.997436	10.99744	11.99744	12.99744

Source: author's computation via SPSS 23

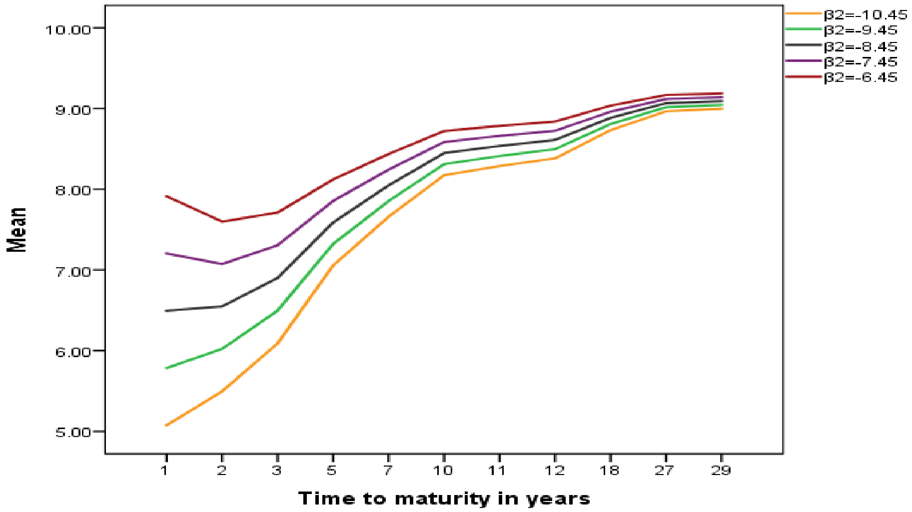

Figure 4: Predicted yields using different values β_2

Table 6: Predicted yields using different values of β_2

Time to maturity	β_2=-10.45	β_2=-9.45	β_2=-8.45	β_2=-7.45	β_2=-6.45
1 year	5.075579	5.785038	6.494497	7.203956	7.913415
2 years	5.496644	6.022182	6.547719	7.073256	7.598794
3 years	6.090609	6.495799	6.900988	7.306177	7.711367
5 years	7.055196	7.321779	7.588362	7.854945	8.121527
7 years	7.65919	7.853493	8.047796	8.242099	8.436402
10 years	8.174346	8.311088	8.447829	8.584571	8.721313
11 years	8.287524	8.411878	8.536232	8.660586	8.784939
12 years	8.382358	8.496368	8.610378	8.724388	8.838398
18 years	8.731875	8.807894	8.883912	8.959931	9.035949
27 years	8.965246	9.015925	9.066604	9.117283	9.167962
29 years	8.997436	9.04462	9.091804	9.138988	9.186172

Source: authors' computation via SPSS 23

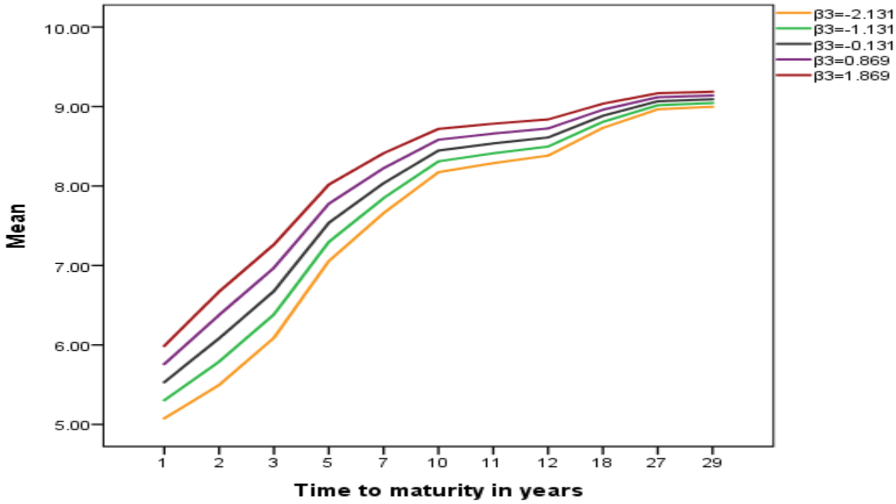

Figure 5: Predicted yields using different values β_3

Table 7: Predicted yields using different values of β_3

Time to maturity	β_3=-2.131	β_3=-1.131	β_3=-0.131	β_3=0.869	β_3=1.869
1 year	5.075579	5.303522	5.531465	5.759409	5.987352
2 years	5.496644	5.790324	6.084004	6.377684	6.671364
3 years	6.090609	6.384155	6.677702	6.971248	7.264795
5 years	7.055196	7.295894	7.536591	7.777289	8.017986
7 years	7.65919	7.847491	8.035793	8.224094	8.412395
10 years	8.174346	8.310418	8.446489	8.582561	8.718633
11 years	8.287524	8.411555	8.535587	8.659618	8.783649
12 years	8.382358	8.496213	8.610068	8.723922	8.837777
18 years	8.731875	8.807892	8.883908	8.959925	9.035941
27 years	8.965246	9.015925	9.066604	9.117283	9.167962
29 years	8.997436	9.04462	9.091804	9.138988	9.186172

Source: author’s computation via SPSS 23

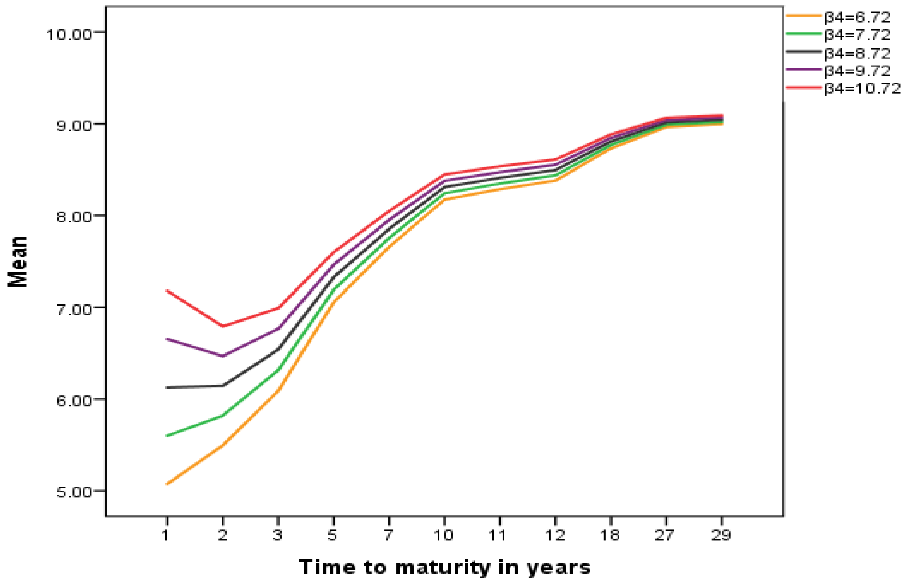

Figure 6: Predicted yields using different values β_4

Table 8 **:** predicted yields using different values of β_4

Time to maturity	β_4=6.72	β_4=7.72	β_4=8.72	β_4=9.72	β_4=10.72
1 year	5.075579	5.601116	6.126654	6.652191	7.177729
2 years	5.496644	5.820338	6.144031	6.467725	6.791419
3 years	6.090609	6.315822	6.541035	6.766248	6.991461
5 years	7.055196	7.191938	7.32868	7.465421	7.602163
7 years	7.65919	7.756925	7.854659	7.952394	8.050128
10 years	8.174346	8.242763	8.311179	8.379596	8.448013
11 years	8.287524	8.349721	8.411918	8.474115	8.536312
12 years	8.382358	8.439372	8.496386	8.5534	8.610414
18 years	8.731875	8.769885	8.807894	8.845903	8.883913
27 years	8.965246	8.990586	9.015925	9.041265	9.066604
29 years	8.997436	9.021028	9.04462	9.068212	9.091804

Source: author's computation via SPSS23

4. Conclusion

Term structure modelling is significant as a result of the opportunities it offers to the investors and regulatory authorities who would apply the results obtained from it to take decision. This paper essentially focuses on estimating the term structure of interest rate of the Nigerian Eurobond for the year 2020. The data obtained for this study was obtained from the debt management office. The data contains 11 maturities (tenors) which include one year, two years, three years, five years, seven years, ten years, eleven years, twelve years, eighteen years, twenty seven years and twenty nine years. The study employed the (Bjork & Christensen, 1999) four-factor model in order to analyze and predict the yields curve and also the yield that were not captured by the observed data. The data obtained from historical daily transaction report of the Nigerian Eurobond was analyzed using descriptive statistics. The study confirms that in the descriptive statistics, the aggregate yield curve is upward sloping. The Bjork and Christensen model four-factor parameters that was analyzed through ordinary least square method after obtaining the value of Lamda enables us to model and predict the in-sample yields. Estimating the Beta's of a Bjork and Christensen's model is conducted by applying a least square methodology and the yield modelling is obtained in accordance with the estimated Beta's**.** Computational evidence from the result reveals a high goodness of fit which was tested using the R square adjusted whose result was 0.97

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